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INTERNATIONAL CONGRESS OF MATHEMATICIANS, 1954.

First Communication.

The International Congress of Mathematicians 1954 will be held in Amsterdam from September 2nd to September 9th under the auspices of "Het Wiskundig Genootschap" (The Mathematical Society of the Netherlands). It is the sincere hope of the "Wiskundig Genootschap" that the Congress 1954, which will be open to all mathematicians from all parts of the world, will be a fertile international gathering.

The Organizing Committee has invited a number of outstanding mathematicians to deliver one-hour addresses, hoping that in this way a survey or the recent development in the whole field of mathematics may be furnished.

There will be seven sections. *viz.*:

1. Algebra and Theory of Numbers.
2. Analysis.
3. Geometry and Topology.
4. Probability and Statistics.
5. Mathematical Physics and Applied Mathematics.
6. Logic and Foundations.
7. Philosophy, History and Education.

Those who wish to attend the Congress are requested to communicate their name (with degrees, qualifications etc.) and full address to the secretariat as soon as possible. They will receive a more detailed communication in due course.

Amsterdam,
2d Boerhaavestraat 49.

The Organizing Committee.

Price per volume Rs. 12/- (India) and Rs. 13/- (Foreign)

ON A GENERALISATION OF BATEMAN'S k-FUNCTION

By

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(*Received—January 10, 1952*)

1. In a paper communicated for publication, Mitra and Srivastava have obtained a generalisation of Bateman's k-function. A similar generalisation has also been obtained by Chakrabarty (1952) in a recent paper published in the Proceedings of the Academy of Sciences, the Netherlands. He has introduced the notation $k^l_n(x)$ which represents the generalised k-function of Bateman, *viz.*,

$$k^l_n(x) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} 2^l \cos^l\theta \cos(x\tan\alpha - n\theta)\,d\theta,\ l > -1.$$

In the two papers mentioned above, various properties of the k^l_n-functions have been obtained and certain relations with the other known special functions have been deduced. In the present paper, further attempts have been made in obtaining a number of properties of this function.

2. Let α and m be positive integers (including zero) and $\alpha \geq m$.

Since $f(p) \doteqdot h(x)$, we have

$$pf(p) \doteqdot \frac{d}{dx}h(x);\ \text{provided}\ h(0) = 0,\ \text{and}\ \frac{p}{(p+1)^{\alpha+1}} \doteqdot \frac{e^{-x}x^{\alpha}}{\Gamma(\alpha+1)}, \tag{1}$$

so,

$$\frac{\Gamma(\alpha+1)p^{m+1}}{(p+1)^{\alpha+1}} \doteqdot \frac{d^m}{dx^m}(e^{-x}x^{\alpha}),\quad \alpha > m. \tag{2}$$

Let us write $m = n-l-1$ and $\alpha = n+l$ and interpret with the help of

$$e^{-\frac{1}{2}x}k^{2l}_{2n}(\tfrac{1}{2}x) \doteqdot (-1)^{n-l-1}\frac{p^{n-l}}{(1+p)^{n-l+1}}; \tag{3}$$

we get

$$e^{-\frac{1}{2}x}k^{2l}_{2n}(\tfrac{1}{2}x) = \frac{(-1)^{n-l-1}}{\Gamma(n+l+1)}\left(\frac{d}{dx}\right)^{n-l-1}(e^{-x}x^{n+l}). \tag{4}$$

In Jacobi's identity

$$\frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)}t^{\alpha}(1-t)^{\beta}{}_2F_1(-n,\ n+1+\alpha+\beta,\ 1+\alpha;\ t) = \frac{d^n}{dt^n}\{t^{n+\alpha}(1-t)^{n+\beta}\},$$

let us write $n = a$, $n+1+\alpha+\beta = b$ and $1+\alpha = c$ and replace t by x/b. Then making $b \to \infty$, we get the relation

$${}_1F_1(-a;c;x) = \frac{\Gamma(c)}{\Gamma(c+a)}e^{x}x^{1-c}\left(\frac{d}{dx}\right)^{a}(e^{-x}x^{c+a-1}). \tag{5}$$

On comparing the right hand sides of (4) and (5) we obtain easily

$$e^{x}k^{2l}_{2n}(x) = \frac{(-1)^{n-l-1}}{\Gamma(2l+2)}(2x)^{2l+1}{}_1F_1(l-n+1,\ 2l+2;\ 2x) \tag{6}$$

and

$$(2x)^{-\alpha} e^{x} k^{\alpha-1}_{2n+\alpha+1}(x) = \frac{(-1)^n n!}{\Gamma(n+\alpha+1)} L^{\alpha}_{n}(2x)\text{[*]}. \tag{7}$$

By using a result due to Howell (1937), *viz.*,

$$L^{\alpha}_{n}(x) = \sum_{r=0}^{n} \frac{\Gamma(\alpha-\beta+r)}{r!\,\Gamma(\alpha-\beta)} L^{\beta}_{n-r}(x)$$

we get, after slight reductions, §

$$x^{-2\alpha-1} e^{\frac{1}{2}x} k^{2\alpha}_{2(n+\alpha+1)}(\tfrac{1}{2}x) = \frac{(-1)^n n!}{\Gamma(n+2\alpha+2)} \sum_{r=0}^{n} \frac{\Gamma(2\alpha-\beta+1+r)}{r!\,\Gamma(2\alpha-\beta+1)} \times L^{\beta}_{n-r}(x). \tag{8}$$

Putting $\beta = \pm\frac{1}{2}$ in sucession, we can deduce the results

$$x^{-2\alpha-\frac{1}{2}} k^{2\alpha}_{2(n+\alpha+1)}(\tfrac{1}{2}x) = \frac{n!}{2^{n+\frac{1}{2}}\Gamma(2\alpha+\frac{1}{2})\Gamma(n+2\alpha+2)} \sum_{r=0}^{n} \frac{(-1)^r 2^r \Gamma(2\alpha+r+\frac{1}{2})}{r!\,(n-r)!} \times D_{2(n-r)+1}(2x)^{\frac{1}{2}} \tag{9}$$

and

$$x^{-2\alpha-1} k^{2\alpha}_{2(n+\alpha+1)}(\tfrac{1}{2}x) = \frac{n!}{2^{n}\Gamma(2\alpha+\frac{3}{2})\Gamma(n+2\alpha+2)} \sum_{r=0}^{n} \frac{(-1)^r 2^r \Gamma(2\alpha+r+\frac{3}{2})}{r!\,(n-r)!} \times D_{2(n-r)}(2x)^{\frac{1}{2}}.$$

whence we obtain the integrals

$$\int_{-\infty}^{\infty} \frac{k^{2\alpha}_{2(n+\alpha+1)}(\frac{1}{2}x^2) D^{(x)}_{2m+1}\,dx}{x^{4\alpha+1}} = \frac{(-1)^{n-m} n!\,(2m+1)!\,\Gamma(2\alpha+n-m+\frac{1}{2})(2\pi)^{\frac{1}{2}}}{2^{2\alpha+m+1}(n-m)!\,m!\,\Gamma(2\alpha+\frac{1}{2})\Gamma(n+2\alpha+2)}, \quad (m \leqq n)$$

$$= 0, \quad m > n \tag{10}$$

and

$$\int_{-\infty}^{\infty} \frac{k^{2\alpha}_{2(n+\alpha+1)}(\frac{1}{2}x^2) D^{(x)}_{2m}}{x^{4\alpha+2}}\,dx = \frac{(-1)^{n-m} n!\,(2m)!\,\Gamma(2\alpha+n-m+\frac{3}{2})(2\pi)^{\frac{1}{2}}}{2^{2\alpha+m+1}(n-m)!\,m!\,\Gamma(2\alpha+\frac{3}{2})\Gamma(n+2\alpha+2)} \quad (m \leqq n)$$

$$= 0 \qquad (m > n). \tag{11}$$

3. We can write $\frac{p}{(p+1)^{n+1}}$ in the form

$$\frac{p}{(p+1)^{n+1}} = \frac{1}{2^l} \sum_{r=0}^{l} \binom{l}{r} \frac{p(1-p)^r}{(1+p)^{n+r+1-l}}$$

and then interpreting term by term with the help of

$$k^{2l}_{2n}(x) \doteqdot 2^{2l+1} \frac{p(1-p)^{n-l-1}}{(1+p)^{n+l+1}},$$

we obtain from (1)

$$\frac{2^n e^{-x} x^n}{\Gamma(n+1)} = \sum_{r=0}^{n-2l-1} \binom{n-2l-1}{r} k^{2l}_{2(r+l+1)}(x). \tag{12}$$

Writing (12) in the form

$$\frac{2^n e^{-2x^2} x^{2n}}{\Gamma(n+1)} = \sum_{r=0}^{n-2l-1} \binom{n-2l-1}{r} e^{-x^2} k^{2l}_{2(r+l+1)}(x^2)$$

and interpreting both sides with the the help of

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[*] This connection of Laguerre polynomial with the Generalized Bateman's Function leads us to surmise that the k^{l}_{m} function can be utilised in the study of wave mechanical treatment of the theory of hydrogen atom, This connection has also been obtained by Mittra and Srivastava.

§ Obtained by Srivastava independently.

$$e^{-2x^2}x^{2n} \doteqdot \frac{\Gamma(2n+1)}{2^{2n+1}}pe^{(1/16)p^2}D_{-2n-1}(\tfrac{1}{2}p) \quad \text{(Howell, 1937).}$$

and $e^{-x^2}k^{2l}_{2n}(x^2) \doteqdot (-1)^{n-l-1}\Gamma(2l+\frac{3}{2})\pi^{-\frac{1}{2}}e^{(1/16)p^2}(\frac{1}{8}p^2)^{\frac{1}{2}n-\frac{1}{2}l-\frac{1}{4}} \times W_{-\frac{1}{2}n-(3/2)l-\frac{3}{4},\ -\frac{1}{2}n+\frac{1}{2}l+\frac{1}{4}}(\frac{1}{8}p^2)$

(Chakrabarty, 1952, Srivastava)

We derive, after slight simplifications

$$pD_{-2n-1}(\tfrac{1}{2}p) = \frac{2^{n+1}\Gamma(n+1)\Gamma(2l+\frac{3}{2})}{\sqrt{\pi}\,\Gamma(2n+1)}\sum_{r=0}^{n-2l-1}(-1)^r\binom{n-2l-1}{r}(\tfrac{1}{8}p^2)^{\frac{1}{2}r+\frac{1}{4}} \times$$
$$\times W_{-\frac{1}{2}r-2l-5/4,\ -\frac{1}{2}r-\frac{1}{4}}(\tfrac{1}{8}p^2) \qquad (13)$$

Again, we have the relation

$e^{-ax} \doteqdot \dfrac{p}{p+a}$ and the well-known formula (Mclachlan and Humbert, 1941) that if $f(p) \doteqdot h(x)$, then

$$h(x^2) \doteqdot \frac{p}{\sqrt{\pi}}\int_0^\infty e^{-\frac{1}{4}p^2s^2}f\left(\frac{1}{s^2}\right)ds.$$

We obtain

$$e^{-ax^2} \doteqdot e^{p^2/8a}\left(\frac{p^2}{4a}\right)^{\frac{1}{4}}W_{-\frac{1}{4},-\frac{1}{4}}\left(\frac{p^2}{4a}\right),\ a > 0. \qquad (14)$$

Let us next write the relation (Chakrabarty)

$$\sum_{m=0}^{\infty} t^m k^{2l}_{2(m-l)}(x) = (1+t)^{2l}e^{x(t-1)/(t+1)},\ |t| < 1$$

in the form

$$\sum_{m=0}^{\infty} t^m e^{-x^2}k^{2l}_{2(m-l)}(x^2) = (1+t)^{2l}e^{-\frac{2}{1+t}x^2}$$

and interpret both sides by known operational representations.

We obtain after slight reductions

$$(1+t)^{2l+\frac{1}{2}}x^{l+\frac{1}{2}}e^{\frac{1}{2}xt}W_{-\frac{1}{4},\ -\frac{1}{4}}\{(1+t)x\}$$
$$= \Gamma(2l+\tfrac{3}{2})\pi^{-\frac{1}{2}}\sum_{m=0}^{\infty}(-1)^{m-1}t^m W_{-\frac{1}{2}m-l-\frac{3}{4},\ -\frac{1}{2}m+l+\frac{1}{4}}(x) \qquad (15)$$

where the left hand side can be expressed in terms of the "Error Function" by the relation

$$\operatorname{erfc}(x) = \tfrac{1}{2}x^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}W_{-\frac{1}{4},+\frac{1}{4}}(x^2)$$

4. Let us now write

$$\frac{2^{l+1}p(1-p)^{\frac{1}{2}n-\frac{1}{2}l-1}}{(1-p)^{\frac{1}{2}n+\frac{1}{2}l+1}} = \frac{2^{l+1}}{p}\cdot p\cdot\frac{(1-p)^{\frac{1}{2}\lambda-\frac{1}{2}\mu-1}}{(1+p)^{\frac{1}{2}\lambda+\frac{1}{2}\mu+1}}\cdot p\cdot\frac{(1-p)^{\frac{1}{2}r-\frac{1}{2}s-1}}{(1+p)^{\frac{1}{2}r+\frac{1}{2}s+1}}$$

where $\lambda + r = n$, and $\mu + s + 2 = l$, we obtain, on interpretation with the help of the product theorem,

$$k^l_n(x) = 2\int_0^x k^\mu_\lambda(x-y)\,k^{l-\mu-2}_{n-\lambda}(y)\,dy \qquad (16)$$

The following particular cases can be deduced at once

$$k^{\mu+2}_n(x) = 2\int_0^r k^\mu_\lambda(x-y)\,k_{n-\lambda}(x)\,dy$$

$$k_n^2(x) = 2\int_0^x k_\lambda(x-y)\,k_{n-\lambda}(y)\,dy$$

$$k_{2n}^{2l}(x) = 2\int_0^x k_n^l(x-y)\,k_n^{l-2}(y)\,dy,\ l > 1. \tag{17}$$

Again we have

$$e^{-\frac{1}{2}x}k_{2n}^{2l}(\tfrac{1}{2}x) \doteqdot (-1)^{n-l-1}\frac{p^{n-l}}{(1+p)^{n+l+1}}$$

and

$$x^{m-\frac{1}{2}}e^{-\frac{1}{2}x}M_{k,m}(x) \doteqdot \frac{\Gamma(2m+1)p^{k-m+\frac{1}{2}}}{(p+1)^{k+m+\frac{1}{2}}}$$

Therefore writing

$$\frac{p^{k-m+\frac{1}{2}}}{(1+p)^{k+m+\frac{1}{2}}} = \frac{1}{p}\cdot\frac{p^{n-l}}{(1+p)^{n+l+1}}\cdot\frac{p^{\mu-\nu+\frac{1}{2}}}{(p+1)^{\mu+\nu+\frac{1}{2}}},$$

where $n+\mu = k$ and $m = l+\nu+1$ and interpreting with the help of the product theorem we obtain

$$x^{m-\frac{1}{2}}M_{k,m}(x) = \frac{(-1)^{n-l-1}\Gamma(2m+1)}{\Gamma(2m-2l-1)}\int_0^x y^{m-l-3/2}k_{2n}^{2l}(\tfrac{1}{2}x-\tfrac{1}{2}y)\times M_{k-n,m-l-1}(y)\,dy \tag{18}$$

Putting $k = n$, $l = 0$ and $m = l+\frac{1}{2}$ and observing that

$$\begin{aligned} M_{n,l+\frac{1}{2}}(x) &= (-1)^{n-l-1}\Gamma(2l+2)k_{2n}^{2l}(\tfrac{1}{2}x), \\ M_{o,n}(x) &= 2^{2n}\Gamma(n+1)x^{\frac{1}{2}}I_n(\tfrac{1}{2}x), \end{aligned}$$

we get

$$\int_0^x y^{l-\frac{1}{2}}k_{2n}(\tfrac{1}{2}x-\tfrac{1}{2}y)I_{l-\frac{1}{2}}(\tfrac{1}{2}y)\,dy = (-1)^l\Gamma(l)\pi^{-\frac{1}{2}}\,k_{2n}^{2l}(\tfrac{1}{2}x),\ l > 0. \tag{19}$$

Putting $k = n$ and $m = l+\frac{3}{2}$ in (18), we also get

$$\int_0^x k_{2n}^{2l}(\tfrac{1}{2}x-\tfrac{1}{2}y)\sinh\tfrac{1}{2}y\,dy = -\tfrac{1}{2}k_{2n}^{2l+2}(\tfrac{1}{2}x) \tag{20}$$

Again writing

$$\frac{p^{n-l}}{(1+p)^{n+l+1}} = \frac{1}{p}\cdot\frac{p^{k-m+\frac{1}{2}}}{(1+p)^{k+m+\frac{1}{2}}}\,\frac{p^{\lambda-\mu+\frac{1}{2}}}{(1+p)^{\lambda+\mu+\frac{1}{2}}},$$

where $k+\lambda = n$, $m+\mu = l$ and proceeding similarly as before, we get

$$\int_0^x y^{l-m-\frac{1}{2}}(x-y)^{m-\frac{1}{2}}M_{k,m}(x-y)M_{n-k,\,l-m}(y)\,dy$$

$$= (-1)^{n-l-1}\Gamma(2m+1)\Gamma(2l-2m+1)\,k_{2n}^{2l}(\tfrac{1}{2}x),\quad \begin{aligned} 2l-2m+1 &> 0, \\ 2m+1 &> 0. \end{aligned} \tag{21}$$

5. From the definition of the Generalised k-function, it is easy to verify that §

$$\frac{d}{dx}k_{2n}^{2l}(x) = k_{2n-1}^{2l-1}(x) - k_{2n+1}^{2l-1}(x) \tag{22}$$

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* This integral has been obtained by Mitra and Srivastava independently.

§ This recurrence relation was also obtained by them.

and
$$\frac{d^2}{dx^2}\,k_{2n}^{2l}(x)=k_{2n}^{2l}(x)-4k_{2n}^{2l-2}(x). \tag{23}$$

We obtain after considerable reductions and making use of the formula

$$e^x k_{2n}^{2l}(x) \doteqdot 2^{2l+1}\frac{(2-p)^{n-l-1}}{p^{n+l}},$$

$$\frac{d^2}{dx^2}\left\{e^{\frac{1}{2}x}k_{2n}^{2l}(\tfrac{1}{2}x)\right\} = e^{\frac{1}{2}x}k_{2(n-1)}^{2(l-1)}(\tfrac{1}{2}x), \qquad (l \geq 1). \tag{24}$$

By the Principle of Induction, we can write

$$D^{2N}\left\{e^{\frac{1}{2}x}k_{2n}^{2l}(\tfrac{1}{2}x)\right\} = e^{\frac{1}{2}x}\,k_{2(n-N)}^{2(l-N)}(\tfrac{1}{2}x). \tag{25}$$

for every positive integral value of N.

We can extend the definition (25) to non-integral values of N *, *viz.*

$$D^{s}\left\{e^{\frac{1}{2}x}k_{2n}^{2l}(\tfrac{1}{2}x)\right\} = e^{\frac{1}{2}x}k_{2(n-\frac{1}{2}s)}^{2(l-\frac{1}{2}s)}(\tfrac{1}{2}x) \tag{26}$$

when s is a positive integer, the expression on the left of (26) represents the s^{th}. differential co-efficient of the expression within the bracket.

6. Let us now consider the product of the two absolutely convergent series

$${}_rF_s\left(\begin{matrix}\alpha_1,\ \alpha_2,\dots\alpha_r\\ \beta_1,\ \beta_2,\dots\beta_s\end{matrix}\ ;\ a^2x^2\right) \equiv {}_rF_s\left(\begin{matrix}\alpha_r\\ \beta_s\end{matrix}\ ;\ a^2x^2\right)$$

$${}_RF_t\left(\begin{matrix}\gamma_1,\ \gamma_2,\ \cdots\ \gamma_R\\ \delta_1,\ \delta_2,\ \dots\ \delta_t\end{matrix}\ ;\ -b^2x^2\right) \equiv {}_RF_t\left(\begin{matrix}\gamma_R\\ \delta_t\end{matrix}\ ;\ -b^2x^2\right)$$

The coefficient of $x^{2\mu}$ in the above product

$$= \sum_{\lambda=0}^{\mu}\frac{\Pi(\alpha_r)_{\mu-\lambda}(-1)^{\lambda}}{(\mu-\lambda)\,!\,\Pi(\beta_s)_{\mu-\lambda}}a^{2\mu-2\lambda}\,b^{2\lambda}\frac{\Pi(\gamma_R)_{\lambda}}{\lambda\,!\,\Pi(\delta_t)_{\lambda}},$$

where
$$\Pi(\alpha r)_{\lambda} \equiv (\alpha_1)_{\lambda}(\alpha_2)_{\lambda}\,..(\alpha_r)_{\lambda}.$$

the above can be written as

$$\frac{a^{2\mu}\Pi(\alpha_r)_{\mu}}{\mu\,!\,\Pi(\beta_s)_{\mu}}\sum_{\lambda=0}^{\mu}\frac{(-\mu)_{\lambda}\,\Pi(-\beta_s-\mu+1)_{\lambda}\,\Pi(\gamma_R)_{\lambda}}{\lambda\,!\,\Pi(-\alpha_r-\mu+1)_{\lambda}\,\Pi(\delta_t)_{\lambda}}\left(\frac{b}{a}\right)^{2\lambda}$$

$$= \frac{a^{2\mu}\,\Pi(\alpha_r)_{\mu}}{\mu\,!\,\Pi(\beta_s)_{\mu}}\,F\left(\begin{matrix}-\mu,\ -\beta_s-\mu+1,\ \gamma_R\\ -\alpha_r-\mu+1\,;\ \delta_t\end{matrix}\ ;\ \frac{b^2}{a^3}\right)$$

Consequently, after some reductions**, we have

$${}_rF_s\left(\begin{matrix}\alpha_r\\ \beta_s\end{matrix}\ ;\ a^2x^2\right)\,{}_RF_t\left(\begin{matrix}\gamma_R\\ \delta_t\end{matrix}\ ;\ -b^2x^2\right)$$

$$= \sum_{\mu=0}^{\infty}\frac{\Pi(\alpha_r)_{\mu}}{\mu\,!\,\Pi(\beta_s)_{\mu}}\,(ax)^{2\mu}F\left(\begin{matrix}-\mu,\ -\beta_s-\mu+1,\ \gamma_R\\ -\alpha_r-\mu-1,\ \delta_t\end{matrix}\ ;\ \frac{b^2}{a^2}\right) \tag{27}$$

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* See Proc Edin. Math. Soc, (2) 6 II (1939) p. 75, a paper by C. G. Lambe or Ferrar—Proc. Roy. Soc. Edin. 48 (1927-28), 92. The argument is on the same lines.

** It is unnecessary to give the details of calculations.

Since

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(\nu+1\,;-\tfrac{1}{4}z^2),$$

we obtain from the above

${}_1F_1(\alpha;\,\beta\,;\,a^2x^2)\,J_\nu\,(bx)$

$$= \frac{1}{\Gamma(\nu+1)}\,(\tfrac{1}{2}bx)^\nu \sum_{\mu=0}^{\infty} \frac{(\alpha)\mu}{\mu\,!\,(\beta)_\mu}\,(ax)^{2\mu}\,{}_2F_2\!\left(\begin{matrix}-\mu,\;-\beta-\mu+1\,; \\ -\alpha-\mu+1,\;\nu+1\,;\end{matrix}\;\left(\frac{b}{2a}\right)^2\right). \tag{28}$$

Put $\alpha = \beta$; we get

$$e^{a^2x^2}J_\nu\,(bx) = \frac{1}{\Gamma(\nu+1)}\,(\tfrac{1}{2}bx)^\nu \sum_{\mu=0}^{\infty} \frac{(ax)^{2\mu}}{\mu\,!}\,{}_1F_1\!\left(-\mu;\;\nu+1\,;\,\frac{b^2}{4a^2}\right) \tag{29}$$

The following particular cases are at once deducible

$$e^{a^2x^2-\frac{b^2}{8a^2}}J_{2l+1}(bx) = \left(\frac{2a^2x}{b}\right)^{2l+1}\sum_{\mu=0}^{\infty}\frac{(-1)^\mu(ax)^{2\mu}}{\mu\,!}\,k^{2l}_{2(\mu+l+1)}\!\left(\frac{b^2}{8a^2}\right), \tag{30}$$

$$e^{a^2x^2-\frac{b^2}{8a^2}}\cos bx = \frac{1}{\sqrt{\pi}}\sum_{\mu=0}^{\infty}\frac{\Gamma(\frac{1}{2}-\mu)}{\mu\,!}\left(\frac{a^2x^2}{2}\right)^\mu D_{2\mu}\!\left(\frac{b}{\sqrt{2a}}\right) \tag{31}$$

$$e^{a^2x^2-\frac{b^2}{8a^2}}\sin bx = \frac{\sqrt{2}ax}{\Gamma(-\frac{1}{2})}\sum_{\mu=0}^{\infty}\frac{\Gamma(-\frac{1}{2}-\mu)}{\mu\,!}(\tfrac{1}{2}a^2x^2)^\mu D_{2\mu+1}\!\left(\frac{b}{\sqrt{2a}}\right) \tag{32}$$

Lastly writing
${}_1F_1(\alpha\,;\,\rho\,;\,\frac{1}{4}x^2)$ in the form

$${}_1F_1\,(\alpha\,;\,\rho\,;\,\tfrac{1}{4}x^2) = \sum_{\mu=0}^{\infty}\frac{(\alpha)_n}{n\,!\,(\rho)_n}(\tfrac{1}{2}x)^{-\mu+n}(\tfrac{1}{2}x)^{\mu+n},$$

and observing that

$$(\tfrac{1}{2}x)^{\mu+n} = \Gamma(\mu+n+1)\sum_{p=0}^{\infty}\frac{(\frac{1}{2}x)^p}{p\,!}J_{\mu+n+p}\,(x), \quad \text{(Watson, 1948)}$$

we obtain, in view of absolute convergence of the repeated series involved (μ being non-negative), the relation

${}_1F_1(\alpha\,;\,\rho\,,\,\frac{1}{4}x^2)$

$$= \sum_{m=0}^{\infty}(\tfrac{1}{2}x)^{-\mu+m}J_{\mu+m}(x)\sum_{n=0}^{\infty}\frac{(\alpha)_n\Gamma(\mu+n+1)}{n\,!\,(m-n)\,!\,(\rho)_n}$$

$$= \sum_{m=0}^{\infty}{}_3F_1\!\left(\begin{matrix}\alpha,\;\mu+1,\;-m \\ \rho\,!\end{matrix}\;,-1\right)\frac{\Gamma(\mu+1)}{m\,!}\,(\tfrac{1}{2}x)^{-\mu+m}J_{\mu+m}(x)\,;$$

the series ${}_3F_1$ being terminating; whence we obtain finally

$e^y(2y)^{\frac{1}{2}\mu-2l-1}k^{2l}_{2n}\,(y)$

$$= (-1)^{n-l-1}\frac{\Gamma(\mu+1)}{\Gamma(2l+2)}\sum_{m=0}^{\infty}{}_3F_1\!\left(\begin{matrix}l-n+1,\;\mu+1,\;-m \\ 2l+2\,!\end{matrix}\;;-1\right)\times\frac{(2y)^{\frac{1}{2}m}J_{\mu+m}(2\sqrt{2y})}{m^4}\,; \tag{33}$$

which is the formula expressing k^l_n in the form of an infinite series involving Bessel functions.

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ON THE BENDING OF AN ELASTIC PLATE—IV

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1. The following pages contain an expression for the deflexion of the central plane of a thin elliptic plate made of isotropic elastic material, clamped at the edge, carrying a load of weight W spread uniformly over a rectangular region on the upper face of the plate. Prior to writing this paper the author had solved the corresponding problem for a circular plate. But when he was about to send his results for publication he received a paper by Prof. Tuyosi Sekiya and Atusi Saito (1949) where they have given a formal expression for the deflexion of the circular plate under arbitrary distribution of load in an integral form and they worked out a particular case where the load is concentrated at a number of discrete points. The author then turned his attention to the problem discussed in this paper. He introduces elliptic co-ordinates for the solution of the problem and ultimately converts the plane harmonic and biharmonic functions appearing in the solution into rational integral functions of x and y thus preparing the results into forms easy for numerical calculation. Biharmonic functions in elliptic co-ordinates appear first to have been introduced by Pöschl (1921) for solving elastic problems with elliptic boundaries. Pöschl (1921) was followed by Happel (1921) and Galerkin (1923) who gave two sets of biharmonic functions in elliptic coordinates suitable for some particular problems which they solved. An exhanstive list of biharmonic functions was given by Timpe which have been used by a large number of workers in the field.

2. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a>b>0)$ be the equation of the edge line of a thin elliptic plate made of isotropic elastic material. Suppose it to be clamped at the edge and let it carry a load of weight W spread uniformly over a rectangular region $X_1<x<X_2$, $Y_1<y<Y_2$ contained entirely within the plate.

As usual, we introduce elliptic co-ordinates by the transformation

$$x+iy=c\cosh(\xi+i\eta) \quad (c>0).$$

Then $\xi=\alpha$, represents the bounding ellipse, provided $a=c\cosh\alpha$ and $b=c\sinh\alpha$.

Let $\xi=\beta$ be an ellipse confocal with the bounding ellipse and lying in its interior, such that the rectangle $X_1 \leqq x \leqq X_2$, $Y_1 \leqq y \leqq Y_2$ is composed entirely of the interior points of the ellipse $\xi=\beta$. Let $(x\ \ y)$ be an arbitrary point in the open region $X_1<x<X_2$ $Y_1<y<Y_2$.

Writing $R^2=(x-x_1)^2+(y-y_1)^2$, we may establish the following result provided K denotes, the rectangle $X_1 \leqq x \leqq X_2$. $Y_1 \leqq y \leqq Y_2$ and $X(n)$ and $Y(n)$ stand for the numbers $\frac{X_2^n-X_1^n}{n}$ and $\frac{Y_2^n-Y_1^n}{n}$ respectively.

$$\iint_K R^2 \log R \, dx_1 \, dy_1 = \frac{a^2-b^2}{4} \sum_{n=0}^{\infty} [f_n(\xi) \cos n\eta + \phi_n(\xi) \sin n\eta] \tag{1}$$

where

$$f_0(\xi) = \xi L_0(\xi) + L_0^{(1)} \exp(2\xi) + L_0^{(2)} + L_0^{(3)} \exp(-2\xi) \tag{2}$$

$$f_1(\xi) = \xi L_1(\xi) + L_1^{(1)} \exp \xi + L_1^{(2)} \exp(-\xi) + L_1^{(3)} \exp(-3\xi) \tag{3}$$

$$\phi_1(\xi) = \xi M_1(\xi) + M_1^{(1)} \exp \xi + M_1^{(2)} \exp(-\xi) + M_1^{(3)} \exp(-3\xi) \tag{4}$$

$$f_2(\xi) = \xi L_2(\xi) + L_2^{(1)} + L_2^{(2)} \exp(-2\xi) + L_2^{(3)} \exp(-4\xi) \tag{5}$$

$$\phi_2(\xi) = \qquad M_2^{(1)} + M_2^{(2)} \exp(-2\xi) + M_2^{(3)} \exp(-4\xi) \tag{6}$$

and for $n \geqq 3$

$$f_n(\xi) = L_n^{(1)} \exp\{-(n-2)\xi\} + L_n^{(2)} \exp(-n\xi) + L_n^{(3)} \exp\{-(n+2)\xi\} \tag{7}$$

$$\phi_n(\xi) = M_n^{(1)} \exp\{-(n-2)\xi\} + M_n^{(2)} \exp(-n\xi) + M_n^{(3)} \exp\{-(n+2)\xi\} \tag{8}$$

and

$$L_0(\xi) = 2X(1)Y(1) \cosh 2\xi + \frac{4}{a^2-b^2} [X(3)Y(1) + X(1)Y(3)] \tag{9}$$

$$L_1(\xi) = -\frac{8X(2)Y(1)}{\sqrt{a^2-b^2}} \cosh \xi \tag{10}$$

$$M_1(\xi) = -\frac{8X(1)Y(2)}{\sqrt{a^2-b^2}} \sinh \xi \tag{11}$$

$$L_2(\xi) = 2X(1)Y(1)$$

$$L_0^{(1)} = \frac{X(1)Y(1)}{2} \log \frac{a^2-b^2}{4} \tag{13}$$

$$L_0^{(2)} = \frac{2}{a^2-b^2}\left(\log \frac{a^2-b^2}{4} - 2\right)[X(3)Y(1) + X(1)Y(3)] \tag{14}$$

$$L_0^{(3)} = \frac{X(1)Y(1)}{2}\left(\log \frac{a^2-b^2}{4} + 2\right) + \frac{2}{a^2-b^2} [X(3)Y(1) - X(1)Y(3)] \tag{15}$$

$$L_1^{(1)} = -\frac{2}{\sqrt{a^2-b^2}}\left(\log \frac{a^2-b^2}{4} + 1\right)X(2)Y(1) \tag{16}$$

$$L_1^{(2)} = -\frac{2}{\sqrt{a^2-b^2}}\left[\left(\log \frac{a^2-b^2}{4} + 2\right)X(2)Y(1) + \frac{2}{a^2-b^2}\{X(4)Y(1) + X(2)Y(3)\}\right] \tag{17}$$

$$L_1^{(3)} = \frac{2}{\sqrt{a^2-b^2}}\left[\frac{2}{3(a^2-b^2)}\{X(4)Y(1) - 3X(2)Y(3)\} - X(2)Y(1)\right] \tag{18}$$

$$M_1^{(1)} = -\frac{2}{\sqrt{a^2-b^2}}\left(\log \frac{a^2-b^2}{4} + 1\right)X(1)Y(2) \tag{19}$$

$$M_1^{(2)} = \frac{2}{\sqrt{a^2-b^2}}\left[\left(\log\frac{a^2-b^2}{4}+2\right)X(1)Y(2)-\frac{2}{a^2-b^2}\{X(3)Y(2)+X(1)Y(4)\}\right] \quad (20)$$

$$M_1^{(3)} = \frac{2}{\sqrt{a^2-b^2}}\left[\frac{2}{3(a^2-b^2)}\{3X(3)Y(2)-X(1)Y(4)\}-X(1)Y(2)\right] \quad (21)$$

$$L_2^{(1)} = \left(\log\frac{a^2-b^2}{4}+1\right)X(1)Y(1)+\frac{2}{a^2-b^2}[X(3)Y(1)-X(1)Y(3)] \quad (22)$$

$$L_2^{(2)} = -\frac{8}{3(a^2-b^2)^2}[X(5)Y(1)-X(1)Y(5)]+\frac{4}{a^2-b^2}[X(3)Y(1)+X(1)Y(3)] \quad (23)$$

$$L_2^{(3)} = \frac{4}{3(a^2-b^2)^2}[X(5)Y(1)-6X(3)Y(3)+X(1)Y(5)] \quad (24)$$

$$-\frac{2}{a^2-b^2}[X(3)Y(1)-X(1)Y(3)]+\tfrac{1}{2}X(1)Y(1) \quad (25)$$

$$M_2^{(1)} = \frac{4}{a^2-b^2}X(2)Y(2)$$

$$M_2^{(2)} = -\frac{16}{3(a^2-b^2)^2}[X(4)Y(2)+X(2)Y(4)] \quad (27)$$

$$M_2^{(3)} = \frac{16}{3(a^2-b^2)^2}[X(4)Y(2)-X(2)Y(4)]-\frac{4}{a^2-b^2}X(2)Y(2) \quad (28)$$

$$L_n^{(1)} = \frac{XY}{(n-1)(a^2-b^2)^{(\frac{1}{2}n-2)}}R\left[\frac{P_n(Z)}{n(a^2-b^2)}-\frac{P_{n-2}(Z)}{n-2}\right] \quad (29)$$

$$L_n^{(2)} = -\frac{2XY}{n(a^2-b^2)^{n/2}}R\left[\bar{Z}\left\{\frac{P_{n+1}(Z)}{(n+1)(a^2-b^2)}-\frac{P_{n+1}(Z)}{n-1}\right\}\right] \quad (30)$$

$$L_n^{(3)} = \frac{XY}{(n+1)(a^2-b^2)^{n/2}}R\left[\frac{P_{n+2}(Z)}{(n+2)(a^2-b^2)}-\frac{P_n(Z)}{n}\right] = L_{n+2}^{(1)} \quad (31)$$

$$M_n^{(1)} = \frac{XY}{(n-1)(a^2-b^2)^{(n-2)/2}}I\left[\frac{P_n(Z)}{n(a^2-b^2)}-\frac{P_{n-2}(Z)}{n-2}\right]. \quad (32)$$

$$M_n^{(2)} = -\frac{2XY}{n(a^2-b^2)^{n/2}}I\left[\bar{Z}\left\{\frac{P_{n+1}(Z)}{(n+1)(a^2-b^2)}-\frac{P_{n-1}(Z)}{n-1}\right\}\right] \quad (33)$$

$$M_n^{(3)} = \frac{XY}{(n+1)(a^2-b^2)^{n/2}}I\left[\frac{P_{n+2}(Z)}{(n+2)(a^2-b^2)}-\frac{P_n(Z)}{n}\right] = M_{n+2}^{(1)} \quad (34)$$

where

$$Z = X+iY,\quad \bar{Z} = X-iY \quad P_n(Z) = (2Z)^n-\frac{n}{1!}(a^2-b^2)(2Z)^{n-2}+\frac{n(n-3)}{2!}(a^2-b^2)^2(2Z)^{n-4}$$

$$-\frac{n(n-4)(n-5)}{3!}(a^2-b^2)^3(2Z)^{n-6}+\ldots\ \ldots \quad (35)$$

The last six of the above formulae are symbolic. The expressions on the right are first to be simplified taking X and Y as algebraic symbols and in the final result X^n and Y^n are to be replaced by $X(n)$ and $Y(n)$ respectively for all positive integral values of n.

In particular $\dfrac{XY}{2(a^2-b^2)^{\frac{1}{2}}}\, R\left[\dfrac{P_3(Z)}{3(a^2-b^2)} - \dfrac{P_1(Z)}{1}\right]$

$$= \frac{2}{\sqrt{a^2-b^2}}\left[\frac{2}{3(a^2-b^2)}\{X^4Y - 3X^2Y^3\} - X^2Y\right]$$

So $L_3^{(1)} = \dfrac{2}{\sqrt{a^2-b^2}}\left[\dfrac{2}{3(a^2-b^2)}\{X(4)Y(1) - 3X(2)Y(3)\} - X(3)Y(1)\right],$

A process of term by term integration has been effected above. The process is valid on account of the uniform convergence of the series involved in $(x_1\, y_1)$ so lond as (x, y) lies in the annular space $\beta < \xi \leqq \alpha$ and $(x_1\, y_1)$ lies in the closed rectangle $X_1 \leqq x \leqq X_2$, $Y_1 \leqq y \leqq Y_2$. It may be proved that $\dfrac{\partial}{\xi}\displaystyle\iint_K R^2 \log R$ may be obtained by term by term differention of the above series for all $\xi > \beta$ and so in particular for $\xi = \alpha$.

3. Let us now consider the expression

$$\omega = -\frac{W}{8\pi DX(1)Y(1)}\iint_K R^2 \log R\, dx_1\, dy_1 + \omega \tag{36}$$

where (Timpe, 1923)

$$\begin{aligned}\omega' = {} & A_0' + A_0 \cosh \xi + (A_1' \cosh \xi + A_1 \cosh 3\xi) \cosh \eta + (B_1' \sinh \xi + B_1 \sinh 3\xi) \sin \eta \\ & + \sum_{n=2}^{\infty} [\{A_{n-2} \cosh (n-2)\xi + A_n' \cosh n\xi + A_n \cosh (n+2)\xi\} \cos n\eta \\ & \qquad + \{B_{n-2} \sinh (n-2)\xi + B_n' \sinh n\xi + B_n \sinh (n+2)\xi\} \sin n\eta]\end{aligned} \tag{37}$$

The boundary conditions for clamped edge, *viz.*, $\omega = \dfrac{\partial\omega}{\partial\xi} = 0$ over $\xi = \alpha$ are formally satisfied if we take

$$A_0' \Delta_0 = Q \begin{vmatrix} f_0(\alpha) & \cosh 2\alpha \\ f_0'(\alpha) & \sinh 2\alpha \end{vmatrix} \tag{38}$$

$$A_0 \Delta_0 = Q \quad f_0'(\alpha) \tag{39}$$

$$A_1' \Delta_1 = Q \begin{vmatrix} f_1(\alpha) & \cosh 3\alpha \\ f_2'(\alpha) & 3 \sinh 3\alpha \end{vmatrix} \tag{40}$$

$$A_1 \Delta_1 = -Q \begin{vmatrix} f_1(\alpha) & \cosh \alpha \\ f_1'(\alpha) & \sinh \alpha \end{vmatrix} \tag{41}$$

$$B_1' \bar{\Delta}_1 = Q \begin{vmatrix} \varphi_1(\alpha) & \sinh 3\alpha \\ \varphi_1'(\alpha) & 3 \cosh 3\alpha \end{vmatrix} \tag{42}$$

$$B_1\,\bar{\Delta}_1 = -Q\begin{vmatrix} \varphi_1(\alpha) & \sinh\alpha \\ \varphi_1'(\alpha) & \cosh\alpha \end{vmatrix} \tag{43}$$

$$A_n'\,\Delta_n = -A_{n-2}(n\sinh 4\alpha + 2\sinh 2n\alpha) + Q\begin{vmatrix} f_n(\alpha) & \cosh(n+2)\alpha \\ f_n'(\alpha) & (n+2)\sinh(n+2)\alpha \end{vmatrix} \tag{44}$$

$$A_n\,\Delta_n = A_{n-2}\,\Delta_{n-2} - Q\begin{vmatrix} f_n(\alpha) & \cosh n\alpha \\ f_n'(\alpha) & n\sinh n\alpha \end{vmatrix} \tag{45}$$

$$B_n'\,\bar{\Delta}_n = B_{n-2}(n\sinh 4\alpha - 2\sinh 2n\alpha) + Q\begin{vmatrix} \varphi_n(\alpha) & \sinh(n+2)\alpha \\ \varphi_n'(\alpha) & (n+2)\cosh(n+2)\alpha \end{vmatrix} \tag{46}$$

$$B_n\bar{\Delta} = B_{n-2}\,\bar{\Delta}_{n-2} - Q\begin{vmatrix} \varphi_n(\alpha) & \sinh n\alpha \\ \varphi_n'(\alpha) & n\cosh n\alpha \end{vmatrix} \tag{47}$$

where

$$Q = \frac{W(a^2-b^2)}{32\pi D X(1) Y(1)} \tag{48}$$

$$\Delta_n = \sinh(2n+2)\alpha + (n+1)\sinh 2\alpha \tag{49}$$

and

$$\bar{\Delta}_n = \sinh(2n+2)\alpha - (n+1)\sinh 2\alpha \tag{50}$$

Explicit formulae for A_n, B_n, A_n' and B_n' may be easily written down.

We may prove that the series $\sum A_n \cosh n\xi \cos n\eta$ is absolutely and uniformly convergent within a closed ellipse that contains the ellipse $\xi = \alpha$ in its interior. It is also possible to show that it is the real part of the series $\sum \frac{A_n}{2c^n} P_n(z)$ where

$$P_n(z) = (2z)^n - \frac{n}{1\,!}(a^2-b^2)(2z)^{n-2} + \frac{n(n-3)}{2\,!}(a^2-b^2)^2(2z)^{n-4} - \frac{n(n-4)\,(n-5)}{3\,!}(a^2-b^2)^3(2z)^{n-6} + \cdots$$

Since the series may be proved to be uniformly convergent in a closed ellipse containing the ellipse $\xi = \alpha$ in its interior, its sum is an analytic function throughout the interior of the bigger ellipse, and denoting the sum by $u(x, y) + iv\,(x, y)$, we have

$$u(x, y) = \sum A_n \cosh n\xi \cos n\eta$$

Since $u(x, y)$ is the real part of a function analytic in the region mentioned above, it is finite and continuous throughout the region and its partial derivatives of all orders with respect to x and y are also finite and continuous throughout the same region. Again $\nabla_1^2 u = 0$ throughout the interior of the region, Without entering into compex mathematical details we shall content ourselves by asserting that the formal expression for ω' given by formula (37) represents a function of x and y finite and continuous throughout the plate including the boundary and that it yields continuous partial derivatives with respect to x and y throughout the same region and that the equation $\nabla_1^4\omega' = 0$ is satisfied throughout the interior of ellipse $0 \leqq \xi < \alpha$. The expressions contributed by ω' to the stress functions G, H and N are therefore finite and continuous throughout the plate including the boundary. It is also possible to demonstrate that the value of $\partial\omega'/\partial\xi$ for $\xi = \alpha$. may be had by term-by-term differentiation of the series for ω' with respect to ξ and then putting $\xi = \alpha$ in the resulting series. This justifies the formal process that has been employed to get the values of the constants A_n, B_n, A_n', B_n'.

4. Now writing

$$U(x, y) = \iint_K R^2 \log R^2 dx_1 dy_1$$

we may shew that

$$\begin{aligned} 3U(x, y) = {} & \mathrm{I}_{1,1}(x, y) + \mathrm{I}_{2,2}(x, y) - \mathrm{I}_{1,2}(x, y) - \mathrm{I}_{2,1}(x, y) \\ & + \tfrac{2}{3}[(X_2 - X_1)\{(y-Y_2)^3 - (y-Y_1)^3\} + (Y_2 - Y_1)\{(x-X_2)^3 - (x-X_1)^3\}] \\ & + (y-Y_2)^4 \left\{\tan^{-1}\frac{x-X_2}{y-Y_2} - \tan^{-1}\frac{x-X_1}{y-Y_2}\right\} \\ & - (y-Y_1)^4 \left\{\tan^{-1}\frac{x-X_2}{y-Y_1} - \tan^{-1}\frac{x-X_1}{y-Y_1}\right\} \\ & + (x-X_2)^4 \left\{\tan^{-1}\frac{y-Y_2}{x-X_2} - \tan^{-1}\frac{y-Y_1}{x-X_2}\right\} \\ & - (x-X_1)^4 \left\{\tan^{-1}\frac{y-Y_2}{x-X_1} - \tan^{-1}\frac{y-Y_1}{x-X_1}\right\} \end{aligned} \tag{51}$$

where
$$\mathrm{I}_{1,1}(x, y) = (x-X_1)(y-Y_1)\{(x-X_1)^2 + (y-Y_1)^2\} \log \{(x-X_1)^2 + (y-Y_1)^2\} \tag{52}$$
$$\mathrm{I}_{2,2}(x, y) = (x-X_2)(y-Y_2)\{(x-X_2)^2 + (y-Y_2)^2\} \log \{(x-X_2)^2 + (y-Y_2)^2\} \tag{53}$$
$$\mathrm{I}_{1,2}(x, y) = (x-X_1)(x-Y_2)\{(x-X_1)^2 + (y-Y_2)^2\} \log \{(x-X_1)^2 + (y-Y_2)^2\} \tag{54}$$
$$\mathrm{I}_{2,1}(x, y) = (x-X_2)(x-Y_1)\{(x-Y_2)^2 + (y-Y_1)^2\} \log \{(x-X_2)^2 + (y-Y_1)^2\} \tag{55}$$

the inverse circular functions involved have their principal values.

We now replace the formula (36) for ω by

$$\omega = -\frac{W}{16\pi D X(1) Y(1)} U(x, y) + \omega' \tag{56}$$

where $U(x, y)$ is given above and ω' is given by the formula (37) of section 3.

$\mathrm{I}_{1,1}(x, y)$ is not defined for (X_1, Y_1), the lower left-hand corner of the rectangle $X_1 \leqq x \leqq X_2$, $Y_1 \leqq y \leqq Y_2$. But it tends to zero as (x, y) tends to (X_1, Y_1). We ascribe to $\mathrm{I}_{1,1}(X_1, Y_1)$ the value zero and similarly we set $\mathrm{I}_{1,2}(X_1, Y_2) = \mathrm{I}_{2,1}(X_2, Y_1) = \mathrm{I}_{2,2}(X_2, Y_2) = 0$. On the other hand the inverse function $\tan^{-1}\frac{x+X_2}{y-Y_2}$ is not defined for $y = Y_2$. It is however bounded throughout the plate. The function $(y-Y_2)^4 \tan^{-1}\frac{x-X_2}{y-Y_2}$ therefore tends to zero as y tends to Y_2. We ascribe to this function the value zero for $y = Y_2$. Similarly we set $(x-X_2)^4 \tan^{-1}\frac{y-Y_2}{x-X_2}$ equal to zero for $x = X_2$. Similar specifications for other terms involving inverse functions are made. This makes $U(x, y)$ and so the function ω finite and continuous throughout the ellipse $0 \leqq \xi \leqq \alpha$ including the boundary. We may shew that derivatives with respect to x and y up to the third order are finite and continuous throughout the plate on our making necessary adjustments for the values of certain functions appearing in them for $x = X_1$, $x = X_2$, $y = Y_1$ and $y = Y_2$ Thus the stresses G, H, N which depend on derivatives of not more than the third order with respect to x ane y are finite and continuous throughout the plate including its boundary.

5. Coming now to the differential equation, the proposed deflexion function ω is expected to satisfy, we can establish that

$$\nabla_1^4 \omega = -\frac{W}{16\pi D X(1) Y(1)} \nabla_1^4 U$$

$$= -\frac{W}{2\pi D X(1) Y(1)} \left[\left(\tan^{-1} \frac{x - X_2}{y - Y_2} + \tan^{-1} \frac{y - Y_2}{x - X_2} \right) \right.$$

$$+ \left(\tan^{-1} \frac{x - X_1}{y - Y_1} + \tan^{-1} \frac{y - Y_1}{x - X_1} \right)$$

$$- \left(\tan^{-1} \frac{x - X_2}{y - Y_1} + \tan^{-1} \frac{y - Y_1}{x - X_2} \right)$$

$$\left. - \left(\tan^{-1} \frac{x - X_1}{y - Y_2} + \tan^{-1} \frac{y - Y_2}{x - X_1} \right) \right] \tag{57}$$

at all points of the plate excepting at those that lie on $x = X_1$, $x = X_2$, $y = Y_1$, $y = Y_2$.

These lines divide the interior of the ellipse into nine compartments. We call them D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8 and D_9

$$D_1 \equiv \underset{x,y}{E}[x > X_2,\ Y_1 < y < Y_2]$$

$$D_2 \equiv \underset{x,y}{E}_{x,y}\,[x > X_2,\ y > Y_2]$$

$$D_3 \equiv \underset{x,y}{E}[X_1 < x < X_2\ \ y > Y_2]$$

$$D_4 \equiv \underset{x,y}{E}[x < X_1,\ y > Y_2]$$

$$D_5 \equiv \underset{x\,y}{E}[x < X_1,\ Y_1 < y < Y_2]$$

$$D_6 \equiv \underset{x,y}{E}[x < X_1,\ y < Y_1]$$

$$D_7 \equiv \underset{x,y}{E}[X_1 < x < X_2,\ y < Y_1]$$

$$D_8 \equiv \underset{x,y}{E}[x > X_2,\ y < Y_1]$$

$$D_9 \equiv \underset{x,y}{E}[X_1 < x < X_2,\ Y_1 < y < Y_2]$$

Evidently D_9 represents the interior of the rectangle $X_1 < x < X_2$, $Y_1 < y < Y_2$.

R_1 therefore contains in its interior an arbitrary piece of the line $y = Y_2$ to the right $x = X_2$.

Similarly let R_2 contain an arbitrary piece of the line $x = X_2$ lying above $y = Y_2$ within it. This will be ensured by $X_1 < \xi_1 < X_2 < \xi_2$ and $Y_2 < \eta_1$
Similarly let

for R_3 $\quad \xi_1 < X_1 < \eta_2 < X_2, \quad Y_2 < \eta_1$
for R_4 $\quad Y_1 < \eta_1 < Y_2 < \eta_2, \quad \xi_2 < X_1$
for R_5 $\quad \eta_1 < Y_1 < \eta_2 < Y_2, \quad \xi_2 < X_1$
for R_6 $\quad \xi_1 < X_1 < \xi_2 < X_2, \quad \eta_2 < Y_1$
for R_7 $\quad X_1 < \xi_1 < X_2 < \xi_2, \quad \eta_2 < Y_1$
and for R_8 $\quad \eta_1 < Y_1 < \eta_2 < Y_2, \quad X_2 < \xi_1$

We shall further consider the rectangles R_9, R_{10}, R_{11} and R_{12} all lying entirely within the plate such that R_9 contains in its interior an arbitrary piece of the line $x = X_2$ lying between $y = Y_1$ and $y = Y_2$. This will be ensured by $X_1 < \xi_1 < X_2 < \xi_2$ and $Y_1 < \eta_1 < \eta_2 < Y_2$ Similarly for R_{10} we take $X_1 < \xi_1 < \xi_2 < X_2$ and $Y_1 < \eta_1 < Y_2 < \eta_2$ for R_{11} we take $\xi_1 < X_1 < \xi_2 < X_2$ and $Y_1 < \eta_1 < \eta_2 < Y_2$ and lastly for R_{12} we have $X_1 < \xi_1 < \xi_2 < X_2$ and $\eta_1 < Y_1 < \eta_2 < Y_2$.

Finally we consider four more rectangles R_{13}, R_{14}, R_{15} and R_{16}, all lying entirely within the ellipse, containing the four corners of the rectangles $X_1 \leqq x \leqq X_2$, $Y_1 \leqq y \leqq Y_2$ in their interiors respectively. We take R_{13} to contain the corner (X_2, Y_2) in its interior. To ensure this, we take

for R_{13}, $\quad X_1 < \xi_1 < X_2 < \xi_2, \quad Y_1 < \eta_1 < Y_2 < \eta_2$
for R_{14}, $\quad \xi_1 < X_1 < \xi_2 < X_2, \quad Y_1 < \eta_1 < Y_2 < \eta_2$
for R_{15}, $\quad \xi_1 < X_1 < \xi_2 < X_2, \quad \eta_1 < Y_1 < \eta_2 < Y_2$
and for R_{16}, $\quad X_1 < \xi_1 < X_2 < \xi_2, \quad \eta_1 < Y_1 < \eta_2 < Y_2$

We now consider the value of the integral $\int_{s_m} N ds$ along the boundaries of the rectangle $R_m (m = 1, 2, 3, \ldots\ldots 16)$ where s_m denotes the boundaries of the rectangle R_m.
By direct integration, we can shew that

$$\int_{s_m} N ds = \frac{W}{2\pi(X_2 - X_1)(Y_2 - Y_1)} I \tag{59}$$

where

$$I = \sum_{p,q,r,s} (-)^{p+q+r+s} (\xi_p - X_q)(\eta_r - Y_s) \left[\tan^{-1} \frac{\xi_p - X_q}{\eta_r - Y_s} + \tan^{-1} \frac{\eta_r - Y_s}{\xi_p - X_q} \right] \tag{60}$$

where each of p, q, r, s assumes the values 1 and 2 and where the summation extends over all possible permutations of 1 and 2 in the four positions p, q, r, s, any integer being permitted to be repeated any number of times.

Now consider the rectangle R_1 where $X_1 < \xi_1 < \xi_2$ and $Y_1 < \eta_1 < Y_2 < \eta_2$.

It is easy to verify that

$$I_{s_1} = \tfrac{1}{2}\pi[(\xi_2-X_2)(\eta_1-Y_2)-(\xi_2-X_1)(\eta_1-Y_2)-(\xi_1-X_2)(\eta_1-Y_2)+(\xi_1-X_1)(\eta_1-Y_2)]$$
$$= 0$$

So

$$\int_{s_1} N ds = 0$$

This proves that there is no concentrated point load or line load along the line $y = Y_2$, $x > X_2$. Similarly we can shew that there is no concentrated point load or a line load along any of the lines $y = Y_2$, $x < X_1$; $y = Y_1$, $x > X_2$; $y = Y_1$, $x < X_1$; $x = X_2$, $y > Y_2$; $x = X_2$, $y < Y_1$; $x = X_1$, $y > Y_2$ and $x = X_1$, $y < Y_1$.

Taking the integral $\int_{s_0} N ds$ along the boundaries of the rectangle R_0 *viz.* $\xi_1 \leqq x \leqq \xi_2$ $\eta_1 \leqq y \leqq \eta_2$ where $X_1 < \xi_1 < X_2 < \xi_2$, $Y_1 < \eta_1 < \eta_2 < Y_2$, we find

$$\int_{s_0} N ds = \frac{W}{(X_2-X_1)(Y_2-Y_1)}(X_2-\xi_1)(\xi_2-\eta_1)$$

Now, since every point in the interior of $X_1 < x < X_2$, $Y_1 < y < Y_2$ is subject to a uniform pressure $W/(X_2-X_1)(Y_2-Y_1)$ and every point of the region $Y_1 < y < Y_2$, $x < X_2$ is free from load and since ξ_1, ξ_2, η_1, η_2 are perfectly arbitrary subject only to the conditions prescribed, it follows that the line $x = X_2$, $Y_1 < y < Y_2$ is free from any point load or a line load. In like manner, we can shew that the lines

$$\begin{array}{ll} x = X_1, & Y_1 < y < Y_2 \\ y = Y_1, & X_1 < x < X_2 \\ y = Y_2, & X_1 < x < X_2 \end{array}$$

are from any concentrated point load or a line load.

We now consider the integral $\int_{s_{13}} N ds$ taken along the boundaries of the rectangle R_{13}, *viz.*, $\xi_1 \leqq x \leqq \xi_2$, $\eta_1 \leqq y \leqq \eta_2$ where $X_1 < \xi_1 < X_2 < \xi_2$, $Y_1 < \eta_1 < Y_2 < \eta_2$, we find

$$\int_{s_{13}} N ds = \frac{W}{(X_2-X_1)(Y_2-Y_1)}(X_2-\xi_1)(Y_2-\eta_1)$$

which considered with the results already obtained proves that the corner (X_2, Y_2) is free from any concentrated point load. In like manner we prove that the other three corners *viz.* (X_2, Y_1), (X_1, Y_2) and (X_1, Y_1) of the rectangle $X_1 \leqq x \leqq X_2$, $Y_1 \leqq y$ Y_2 are also free from any concentrated point load.

We have therefore established that the function

$$\omega = -\frac{W}{16\pi D(X_2-X_1)(Y_2-Y_1)}U(x, y)+\omega'$$

where $U(x, y)$ is given by (51) and ω' is given by (37) and the constants A_n, A_n', B_n, B_n' are given by the formulae (38), (39),......(50) read along with the formulae (2), (3),......(35) represents the deflexion of the central plane of a thin elliptic plate clamped at the edge

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ carrying a load of weight W spread uniformly over the rectangle $X_1 < x < X_2$, $Y_1 < y < Y_2$ on the upper face of the plate.

Note.—It may be mentioned here that the deflexion of the central plane of a thin elliptic plate clamped at the edge $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, carrying a load of weight W along a line parallel to the minor axis, say along $x=X_1$, $Y_1 < y < Y_1$ may be written down from the expression giving the deflexion in this paper by making $X_2 \rightarrow X_1$. The limiting function obtained by effecting the above process, term by term, may be shown to satisfy all the requirements of the problem.

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ON THE GENERAL THEORY OF DOWNWASH BEHIND A FINITE WING IN SUPERSONIC RANGE

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Introduction

The purpose of the present paper is to derive the most general theory of the downwash phenomenon behind a finite wing in a nonviscous isentropic fluid. The integro-differential equation obtained in the last step of the theory is solved by means of Picard's iteration process. As is known, there are four items which may cause discrepancies between the numerical results based on the linearized theory and the experimental data: distortion and displacement of the vorticity sheet, volume distribution of vorticity and influence of viscosity. The first two items jointly with the influence of a high angle of attack are taken into account in the present note. The third item seems to be practically of small importance and the fourth one has not yet been determined. The author assumes that the reader is acquainted with the papers of previous writers on the subject, and consequently, only a few items will be cited from the previous literature.

Part I. Basic Notions and Equations.

1.1. Potential Equation and Its Transformation.

Assume a steady, irrotational, three-dimensional flow of a nonviscous, isentropic fluid past a body. The velocity potential equation has the form:

$$c^{-2}\overset{1}{\varphi}_{,i}\,\overset{1}{\varphi}_{,k}\,\overset{1}{\varphi}_{,ik} = \overset{1}{\varphi}_{,jj}, \quad \overset{1}{u}_i = \overset{1}{\varphi}_{,i}, \tag{1.1.1}$$

where the ordinary tensor notation is used and the system of coordinates (x_1, y_1, z_1) is located as follows. the x_1-axis is in the direction of the free-stream flow, the $x_{12} = y_1$-axis lies in the horizontal plane perpendicular to x_1-axis and the $x_{13} = z_1$-axis is vertical upwards. Let us decompose the velocity components in the sense that $\overset{1}{u}_1 = U+\varphi_{,x_1} = U+u_1 = U+u$, $\overset{1}{u}_i = \overset{1}{\varphi}_{,x_i} = \varphi_{,x_i} = u_i$, $i = 2, 3$, $u_2 = v$, $u_3 = w$. From the energy equation $c^2+\beta V^2 = c_\infty^2+\beta V_\infty^2$, $\beta = \frac{1}{2}(\gamma-1)$, one may calculate

$$c^{-2} = c_\infty^{-2}\left[1+Z_1(1-Z_1)^{-1}\right], \quad Z_1 = \beta c_\infty^{-2} Y_1, \tag{1.1.2}$$

where

$$Y_1 = V^2-U^2 = 2Uu_1+\textstyle\sum_1^3(u_i)^2 = 2U\varphi_{,x_1}+\sum_1^3(\varphi_{,i})^2.$$

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Introduce the affine transformation:

$$x = x_1, \; y = \mu y_1, \; z = \mu z_1, \; \mu = \left[-(1-M_\infty^2)\right]^{1/2}.$$

Combining (1.1.1) and (1.1.2) results in:

$$\varphi_{,xx}-\varphi_{,yy}-\varphi_{,zz} \equiv \Box^2\varphi = F(\varphi(x,\ y,\ z)) = G(x,\ y,\ z), \tag{1.1.3}$$

$$F(\varphi) = [(1-M_\infty^2)(c_\infty^2-\beta Y_2)]^{-1}\textstyle\sum_1^2 G_i,\quad G_1 = \beta M_\infty^2 Y_2\varphi_{,xx},$$

$$Y_2 = 2U\varphi_{,x}+(\varphi_{,x})^2+\mu^2\textstyle\sum_y^z(\varphi_{,i})^2,\quad G_2 = 2U\epsilon\varphi_{,i}\varphi_{,xi}+\eta\varphi_{,i}\varphi_{,k}\varphi_{,ik},$$

where i, j, k denote now the coordinates x, y, z and the quantities ϵ and η are:

$$\epsilon = \begin{cases} 1 \text{ if } i = x, \\ \mu^2 \text{ if } i = y \text{ or } z, \end{cases} \qquad \eta = \begin{cases} 1 \text{ if } i = k = x, \\ \mu^2 \text{ if } i = x,\ k = y \text{ or } z, \text{ if } i = y \text{ or } z,\ k = x, \\ \mu^4 \text{ if } i = y \text{ or } z,\ k = y \text{ or } z. \end{cases}$$

For some future purposes introduce the operator $\Box\varphi \equiv \varphi_{,x}-\varphi_{,y}-\varphi_{,z}$, which applied to a scalar is analogous to a gradient and applied to a vector is analogous to a divergence. Fundamental solution associated with (1.1.3) is

$$\varphi = \iiint_K r^{-1}G\,d\xi\,d\eta\,d\zeta,$$

where $r^2 = (x-x_2)^2-(y-y_2)^2-(z-z_2)^2$, and K is the domain in $(x_2,\ y_2,\ z_2)$-space defined by the inequalities $x_2 \leqq x$, $(x-x_2)^2 \geqq (y-y_2)^2+(z-z_2)^2$. This solution represents the potential at $P(x,\ y,\ z)$ of the space-$(x_2,\ y_2,\ z_2)$-mass distribution of the variable density G. In case $\Box^2\varphi = 0$, the fundamental solution is $\varphi = r^{-1}$. Of course, the variables of integration are the $(x_2,\ y_2,\ z_2)$-coordinates.

1.2. Modified Green's and Gauss's Theorems.

If n denotes the inner normal to S_2 of D in a three dimensional space, then Green's second formula is:

$$\iiint_D [u\nabla^2 v - v\nabla^2 u]\,dx_2dy_2dz_2 = -\iint_{S_2} [uv_{,n}-vu_{,n}]\,dS_2. \tag{1.2.1}$$

The co-normal ν to S_2 is defined by $\nu_1 = -n_1 = \cos(\pi-\alpha) = -\cos\alpha$, $\nu_2 = n_2 = \cos\beta$, $\nu_3 = n_3 = \cos\gamma$, where n_i, $i = 1$ to 3, are direction cosines of n. Repeating the procedure applied to derive Green's second formula, one easily obtains "modified" Green's formula (Heaslet and Lomax, 1948, *a*, *b*):

$$\iiint_D [u\Box^2 v - v\Box^2 u]\,dx_2dy_2dz_2 = \iint_S [uv_{,\nu}-vu_{,\nu}]\,dS_2. \tag{1.2.2}$$

Thus, the operator ∇^2 transforms into $-\Box^2$ and $\partial/\partial_n \rightarrow \partial/\partial\nu$.

In case $v = r^{-1}$, there is a singularity at P. On the right side in (1.2.1) one must add the term pu, where $p = 4\pi$, 2π, or 0 for P inside D, on S_2 or outside D, respectively. An analogous situation is in (1.2.2), *i.e.*, on the right-hand side of (1.2.2) one has to add the term $-pu$. The sign $-$ is due to the fact that passing from (1.2.1) to (1.2.2) we multiply the entire equation by -1. Let $v = r^{-1}$, or $\Box^2 v = \Box^2(r^{-1})$, $u = \varphi$, then (1.2.2) transforms into:

$$p\varphi = -\iint\limits_{S_2}[r^{-1}\varphi_{,\nu} - \varphi(r^{-1})_{,\nu}]dS_2 + \iiint\limits_{D} r^{-1}G\,dD. \tag{1.2.3}$$

Let in (1.2.1) $u = 1$, $\nabla^2 v \equiv \operatorname{div}\operatorname{grad} v = \operatorname{div}\vec{a}$, or $\vec{a} = \operatorname{grad} v$, and $\vec{a} = \vec{i}a + \vec{j}a + \vec{k}a$, $|\vec{a}| = 3^{1/2}a$. Hence, $v_{,n} = \operatorname{grad}_n v = a_n = \vec{a}.\vec{n}$. Then from (1.2.1) one gets Gauss's theorem:

$$\iiint\limits_{D}\operatorname{div}\vec{a}\,dD = -\iint\limits_{S_2} a_n dS_2. \tag{1.2.4}$$

The right-hand side of (1.2.4) may be written in the form

$$-\iint\limits_{S_2} a(\cos\alpha + \cos\beta + \cos\gamma)dS_2.$$

Similarly, put in (1 2.2) $u = 1$, $\Box^2 v \equiv \Box.\operatorname{grad} v = \Box\vec{a}$, and obtain the "modified" Gauss theorem:

$$\iiint\limits_{D}\Box\vec{a}\,dD = \iint\limits_{S_2} a_\nu dS_2. \tag{1.2.5}$$

As is seen, $\operatorname{div}\vec{a} \rightarrow -\Box\vec{a}$ and $\vec{a}.\vec{n} \rightarrow \vec{a}.\vec{\nu}$. Transform the second integral in (1.2.3) by means of the modified Gauss theorem. Evidently it must be:

$$\Box\vec{a} = a_{,x_2} - a_{,y_2} - a_{,z_2} = r^{-1}G = M(x, y, z; x_2, y_2, z_2).$$

This is Lagrange's linear equation whose subsidiary equations (equations of characteristics) are:

$$dx_2/1 = dy_2/-1 = dz_2/-1 = da/M.$$

Applying a set of multipliers one gets:

$$da = \tfrac{1}{3}M(dx_2 - dy_2 - dz_2).$$

from which it is obvious (since $da = a_{,x_2}dx_2 + a_{,y_2}dy_2 + a_{,z_2}dz_2$) that $a_{,x_2} = \frac{1}{3}M$, $a_{,y_2} = a_{,z_2} = -\frac{1}{3}M$, and hence,

$$a = \frac{1}{3}\left\{\int_{x_2} Mdx_2 - \int_{y_2}^{y} Mdy_2 - \int_{z_2}^{z} Mdz_2\right\}, \tag{1.2.6}$$

where the integrals extend from a point (x_2, y_2, z_2) on S_2 to the point $P(x, y, z)$ in K or D. Thus, the final form of (1.2.3) is

$$p\varphi = -\iint\limits_{S_2}[r^{-1}\varphi_{,\nu} - \varphi(r^{-1})_{,\nu} - a_\nu]\,dS_2, \tag{1.2.7}$$

with $a_\nu = a(-\cos\alpha + \cos\beta + \cos\gamma) = a\bar{h}_2$. Since the function a has a singularity at P, one has to investigate it in the neighborhood of P. Assume a sphere C_ϵ of radius ϵ around P as the origin.

Thus,

$$\iint\limits_{S_1} = \iint\limits_{S_2} + \iint\limits_{C_\epsilon}.$$

Since $dC_\epsilon = \epsilon^2 \sin\varphi d\varphi d\theta$, the last integral in (1.2.7) takes the form :

$$\lim_{\epsilon\to 0}\int_0^{x_1,\, y_1,\, z_1 \text{ on } C_\epsilon} (dx_2 - dy_2 - dz_2)\int_0^{2\pi} d\theta \int_0^{\pi} \epsilon G h_2 \sin\varphi d\varphi$$
$$= \lim_{\epsilon\to 0} 4\pi\epsilon (G h_1 h_2)_{x_1,\, y_1,\, z_1 \text{ on } C_\epsilon} = 0,$$

with $h_1 = (x_2 - y_2 - z_2)$. In all the cases (P inside K, on S_2 or outside K) the integration over C_ϵ yields zero, and there is no effect of the singularity at P (due to a_ν) upon the form of the integro-differential equation (1.2.7). In the linearized theory $a_\nu \equiv 0$.

1.3. Picard's Iteration Process.

The solution of the integro-differential equation (1.2.7) will be achieved by Picard's iteration procedure. The most general form of (1.2.7) is

$$\varphi = \iint\limits_{S_1} F dS_2, \quad F = F(x_i, h_2, r^{-1}, \varphi, \varphi_{,\nu}, \varphi_{,i}, \varphi_{,ik}).$$

The proof of convergence of the iteration process is the same as in (Courant and Hilbert, 1937) or (Tamarkin and Feller, 1941). Consequently, only a few deviations from the pattern given in (Tamarkin and Feller, 1941) will be marked. The first part of the proof is unchanged with $F_0 = F_0(x_i\ \ h_2, r^{-1}, f^{(u)}, f^{(l)}, \varphi_0, \varphi_{0,\nu}, \varphi_{0,i}, \varphi_{0,ik})$, φ_0 being an identically zero function and $f^{(u)}$, $f^{(l)}$, denoting certain given initial function of φ or $\varphi_{,\nu}$, which should be used only once at the beginning of the procedure. Instead of (8.40) in (Tamarkin and Feller, 1941) one has the relations :

$$\varphi_n = \iint\limits_{S_1} F_{n-1} dS_2, \quad \varphi_{n,\nu} \equiv p_n = \iint\limits_{S_1} F_{n-1,\nu} dS_2, \tag{1.3.1}$$

$$\varphi_{n,i} \equiv q_n = \iint\limits_{S_2} F_{n-1,i} dS_2, \quad \varphi_{n,ik} \equiv s_n = \iint\limits_{S_2} F_{n-1,ik} dS_2.$$

Assume that there exists a number $K > 0$ such that for all x_i, h_2, r^{-1}, φ, etc., for which $\max(|\varphi|, |p|, |q|, \ldots, |f^{(u)}|, \ldots) < K$, $F_{,\nu}$, $F_{,i}$, $F_{,\varphi}$, . . , exist and $\max(|F|, |F_{,\nu}|, \ldots, |\tilde{F}_{,\varphi}|, \ldots) < M$. With $\max(|\varphi_{n-1}|, |p_{n-1}|, |q_{n-1}, \ldots) < K$, one obtains :

$$|\varphi_n| < Ml^2, \ |p_n| < Ml^2, \ |q_n| < Ml^2, \ |s_n| < Ml^2.$$

With $Ml^2 < K$, we can be sure that $\varphi_n, \ldots,$ are defined. Similarly, as in (Tamarkin and Feller, 1941) apply the law of the mean and with

$$I_s = \iint\limits_{S_1} (|\varphi_n - \varphi_{n-1}| + |p_n - p_{n-1}| + |q_n - q_{n-1}| + |s_n - s_{n-1}|) dS_2,$$

and $F_{,\nu} = pF_{,\varphi}$, $F_{,ik} = F_{,\varphi\varphi} q_i q_k + F_{,\varphi} s_{ik}$, etc., obtain :

$$|\varphi_{n+1} - \varphi_n| < MI_s, \ |p_{n+1} - p_n| < M^2 l^2 I_s, \ |q_{n+1} - q_n| < M^2 l^2 I_s,$$

$$|s_{n+1}-s_n| < M^2l^2(1+Ml^2)I_s. \qquad (1.3.3)$$

Denoting by S_n the maximum value the integrand of I_s can take we obtain as the sum of the absolute values in (1.3.3) the expression:

$$Ml^2(1+3Ml^2+M^2l^4)S_n = \alpha S_n. \qquad (1.3.4)$$

The last part of the proof of convergence with $\alpha < 1$ and the proof of uniqueness are identically the same as in (Courant and Hilbert, 1937) or (Tamarkin and Feller, 1941). In order that the solution exists, the condition $\alpha < 1$ must be preserved. This holds for any M if $S_2 \sim l^2$ is sufficiently small, or for any S_2 if M is sufficiently small. This latter occurs in the method of small perturbations.

1.4. Transformation of Multiple Integrals.

Let the transformation $x = \varphi(u, v)$, $y = \psi(u, v)$ represent a continuous one-to-one mapping of the closed region R of the xy-plane on a region R' of the uv-plane. Let the function $z = f(x, y) = \chi(u, v)$ represent a surface which lies above the region R (or R') of the xy (or uv)-plane. Let all the functions φ, ψ, z, χ possess continuous first derivatives and let the Jacobian $D_1 = \partial(x, y)/\partial(u, v)$ be everywhere positive, whereas Jacobians: $D_2 = \partial(\psi, \chi)/\partial(u, v)$, and $D_3 = \partial(\chi, \varphi)/\partial(u, v)$ may have any values or may even vanish. Then, as is known, (Courant, 1937), the area of a curved surface is given in the (x, y)-system by the formula

$$A = \iint_R [1+(f_{,x})^2+(f_{,y})^2]^{1/2}dS, \qquad (1.4.1)$$

with $dS = dxdy$ and R representing the projection of the surface on the xy-plane, and in the (u, v)-system by the formula

$$A = \iint_{R'} [\textstyle\sum_{i=1}^{3}(D_i)^2]^{1/2}dS', \qquad (1.4.2)$$

with $dS' = dudv$ and R' representing the projection of the surface on the uv-plane. A double integral over a curved surface z, $\iint_z FdS$ may be represented either by means of (1.4.1) as $\iint_R F[1+\ldots]^{1/2}dxdy$ in the (x, y)-system or by means of (1.4.2) as $\iint_{R'} F[\sum\ldots]^{1/2}dudv$ in the (u, v)-system.

1.5. Hadamard's Improper Integral.

The integral

$$I = \int_a^b \frac{A(x)}{(b-x)^{p+1/2}}\,dx, \qquad (1.5.1)$$

is meaningless (p being any integer). To have somehow a means of calculating the integral (1.5.1) Hadamard (1923) introduced a new kind of improper integral (the "finite part" of the integral in question with the symbol $\ulcorner$), Without going into too many details, we cite here only the final result.

$$\overline{\Big|\int_a^b}\frac{A(x)dx}{(b-x)^{p+1/2}}=-\frac{A(b)}{(p-\frac{1}{2})(b-a)^{p-1/2}}+\ldots-\frac{(-1)^{p-1}A^{(p-1)}(b)}{(p-1)!\frac{1}{2}(b-a)^{1/2}}+\int_a^b\frac{A_1(x)dx}{(b-x)^{p+1/2}},\quad(1.2.5)$$

where

$$A_1(x)=A(x)-[A(b)-A'(b)(b-x)+\ldots+(-1)^{p-1}A^{(p-1)}(b)(b-x)^{p-1}].\quad(1.5.2a)$$

One may introduce another representation of a finite part of an improper integral (Hadamard, 1923). The indefinite integral (1.5.1) with $p=1$, say, can be written in the form $F(b)+C-F(a)-C$. Develop the function $A(x)$ in Taylor's series, i.e., $A(x)=A(b)+(x-b)A'(b)+\ldots$, and calculate the first integral of the expansion; i.e.,

$$I_3=\int_a^b\frac{A(b)dx}{(b-x)^{3/2}}=\frac{2A(b)}{(b-x)^{1/2}}\Big|_a^b.\quad(1.5.3)$$

Inserting the upper limit b into the integral (1.5 3) gives a meaningless value. One can show that such a case occurs only for the first term of the expansion. Hence, in order to obtain a finite value of that integral one must subtract that expression from the original value of the integral. In other words,

$$I_3=\overline{\Big|\int_a^b}\frac{A(x)dx}{(b-x)^{3/2}}=\lim_{x\to b}\left[[F(x)+C-F(a)-C-\frac{2A(b)}{(b-x)^{1/2}}\right].\quad(1.5.4)$$

It follows that if C is chosen so that

$$C=\lim_{x\to b}\left[\frac{2A(b)}{(b-x)^{1/2}}-F(x)\right],$$

then the expression (1.5.4) for I_3 may be written

$$I_3=-[F(a)+C].$$

When C is chosen in this manner, the notation for the calculation may thus be modified to the form

$$I_3=\int_a^{*}\frac{A(x)dx}{(b-x)^{3/2}}=-[F(a)+C],$$

where the asterisk indicates that the upper limit is not substituted into the indefinite integral, $F(x)+C$, C being calculated as explained above.

PART II. WING THEORY AND VORTICITY SHEET.

2.1. **Pressure.**

Assume a supersonic flow past a lifting surface of symmetrical planform. The plane of symmetry lies in the x_1z_1-plane, with $+x_1$ directed downstream, $+y_1$ sideward and $+z_1$ upward. From the Bernoulli's equation, one gets

$$p = p - p_\infty = -\int_{V_\infty}^{V} \varrho V dV.$$

Since $Y_1 = V^2 - U^2$, $VdV = \frac{1}{2} dY_1$. With $\varrho = \varrho_\infty (1 - Z_1)^{1/(2\beta)}$. one may solve the integral in question by elementary method obtaining:

$$p = H(Y_1) = \varrho_\infty c_\infty^2 \gamma^{-1} [(1 - Z_1)^\epsilon - 1], \quad \epsilon = \gamma(\gamma - 1)^{-1}. \tag{2.1.1}$$

Let the subscripts u, l, denote the conditions on the upper and lower parts of the lifting surface, respectively, then, with $q = \frac{1}{2} \varrho_\infty U^2$.

$$\Delta p/q = (\bar{p}_l - \bar{p}_u) q^{-1} = (p_l - p_u) q^{-1} = \varrho_\infty c_\infty^2 (\gamma q)^{-1} [(1 - Z_{1l})^\epsilon - (1 - Z_{1u})^\epsilon]. \tag{2.1.2}$$

From (2.1.1), one easily obtains·

$$u_1 = -U[1 - (1 - U^{-2} B)^{1/2}], \quad B = \varphi_{,y_1}^{2} + \varphi_{,z_1}^{2} - c_\infty^2 \beta^{-1} [1 - (1 + \bar{p} \gamma \varrho_\infty^{-1} c_\infty^{-2})^{1/\epsilon}]. \tag{2.1.3}$$

Applying twice the binomial theorem in (2 1 3) and preserving only one term, one obtains $\bar{p}/q = -2U^{-1} u_1$, *a* known quantity from the linearized theory.

2.2. Supersonic Wing Theory

Assume, for the sake of simplicity, that a supersonic wing is substituted by a flat plate of zero thickness with the trailing edge parallel to y_1-axis. Denote the angle of attack by δ_1 and introduce the additional system of coordinates (t_1, y_1, n_1) where t_1 denotes the tangent and n_1 the normal to the plate respectively. Then,

$$t_1 = x_1 \cos \delta_1 - z_1 \sin \delta_1, \quad n_1 = x_1 \sin \delta_1 + z_1 \cos \delta_1,$$

and

$$u_t^1 = u_1^1 \cos \delta_1 - u_3^1 \sin \delta_1, \quad u_n^1 = u_1^1 \sin \delta_1 + u_3^1 \cos \delta_1. \tag{2.2.1}$$

With

$$u_t^1 = U \cos \delta_1 + u_t, \quad u_n^1 U \sin \delta_1 + u_n,$$

the components u_t and u_n may be easily calculated. Moreover, it is obvious that

$$V^2 = (u_t^1)^2 + v^2 + (u_n^1)^2.$$

The following conditions are superimposed upon the system: the u_t-and $u_2 = v$-velocity components are antisymmetric and the u_n-component is symmetric about the plane of the lifting surface. Thus, the velocity components on the top and bottom surface of the lifting surface are related by:

$$u_{tl} = -u_{tu}, \quad v_l = -v_u, \quad u_{nl} = u_{nu},$$

as $n \to$ surface. The discontinuities in u_t and v constitute a vorticity sheet. Preserving all the notions and definitions introduced into the supersonic wing theory by previous authors, we obtain the velocity potential from the integral

$$\varphi = \int \vec{V} \cdot d\vec{s},$$

and the circulation from $\Gamma = \int \vec{V} \cdot d\vec{s}$, the integral being taken round the circuit

where
$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w,$$

and
$$\vec{s} = \vec{i}x_1 + \vec{j}y_1 + \vec{k}z_1.$$

Assuming that the circulation is equal to the jump in φ when approaching to the lifting surface, one obtains:

$$\Gamma \equiv \Delta\varphi_s = \int_{x_{1L}}^{x_{1T}} (u_u - u_l)dx_1 + \int_{y_{1L}}^{y_{1T}} (v_u - v_l)dy_1 + \int_{z_{1L}}^{z_{1T}} (w_u - w_l)dz_1, \tag{2.2.2}$$

where the subscripts L and T denote the points on the leading and trailing edge, respectively. Equation (2.2.2) can be written as

$$\Delta\varphi_s = \int_0 \operatorname{grad}(\varphi_u - \varphi_l) \cdot d\vec{s}.$$

To obtain the circulation per unit span, one has to integrate along the line $y_1 = \text{const.}$, *i.e.*, this is given by (2.2.2) without the second integral. For comparison purposes, one may reduce the conditions to a linearized flow: $\delta_1 \approx 0^0$, $n_1 = z_1$, $w_u = w_l$, and $\Delta\varphi_s$ is given by the first integral in (2.2.2). Preserving only one term in (2.1.3), the difference $(u_u - u_l)$ is equal to $U(2q)^{-1}(p_l - p_u)$, which is known from the linearized theory. To have a lift, there must be: $\bar{p}_l > 0$, $\bar{p}_u < 0$, i.e., the function $H(Y_1)$ (2.1.1) is an odd function of n_1, since it changes its sign when approaching to the lifting surface in the n_1-direction.

2.3. Vorticity Sheet.

The vorticity system behind a lifting surface consists of a vorticity sheet and two vortex cores except just at the trailing edge, where it is only a vorticity sheet and no cores, and at infinity, where there are only two cores and no vorticity sheet. Although the fluid is nonviscous, let us assume a transfer of vorticity from the center part of the vorticity sheet towards the cores. The total strength Γ_0 of the vortex-lines trailing from each half of the lifting surface remains constant at all stations. Following (Spreiter and Sacks, 1951) assume that the strength of the vortex-lines Γ_w between two cores is given by

$$\Gamma_w(x_1, y_1) = \Gamma(x_{1t}, y_{1t})P(x), \quad 0 \leqq P \leqq 1,$$

where $\Gamma(x_{1t}, y_{1t})$ refer to the trailing edge. Hence,

$$\Gamma_0 = \int_0^{y_{1c}} \Gamma_w dy_1 + \Gamma_c,$$

where y_{1c} and Γ_c denote the variable lateral position of the center and the strength of a core, respectively. Since

$$\int_0^{y_{1c}} \Gamma dy_1 = \Gamma_0 - \Gamma,$$

one has $\Gamma_0 = \Gamma_c + P(\Gamma_0 - \Gamma)$. For $x_1 \rightarrow \infty$, the semi-span s_1 of completely rolled-up cores is given in (Spreiter and Sacks, 1951) and is equal to $\frac{1}{2} C_L S (c_l c)^{-1}_{y_1=0}$, where C_L is the lift coefficient of the lifting surface of area S, and $(c_l c)_{y_1=0}$ is the product of the section lift coefficient and the chord at $y_1 = 0$. Similarly, the function $P(x_1)$ is calculated in (Spreiter and Sacks, 1951) and is equal to:

$$P(x_1) = [y_{1o}(x_1) - s'][y_{1c}(x_1) - \int_0^{y_{1c}} (\Gamma/\Gamma_0) dy_1]^{-1}. \tag{2.3.1}$$

An approximate determination of the diameter of the core, as given in (Spreiter and Sacks, 1951) may be preserved in the present case with two changes: instead of ϱ_0, the quantity ϱ_c, and instead of s', the quantity y_{1o} should be used, respectively, where ϱ_c may be assumed to refer to the center of the core. Assuming that the diameter a_c of the core may not be much less than s_1' we may calculate a_c from the original equation in (Spreiter and Sacks, 1951) instead of from the simplified equation, thus obtaining:

$$a_c (s_1')^{-1} = 2 \exp \alpha [\exp(2\alpha) - 1]^{-1}, \tag{2.3.2}$$

$$\alpha = 16\pi \varrho_0 s_1'^2 c_{D_i} (\varrho_c S C_L^2)^{-1} - 1/4. \tag{2.3.2a}$$

Preserving the analogue with the lifting surface, let it be assumed that behind the surface one has: $\Delta\varphi_{sw} = \Gamma_w$ or Γ_c. To find the shape of the vorticity sheet at any distance behind the trailing edge, we have to calculate the down-and side-displacement of a vortex-line. Similarly, to know the value of $P(x_1)$ (2.3.1), we have to know the function y_{1o}, which refers to the vortex-line originated at the tip and next displaced in both vertical and transverse directions.

Any displacement in y_1-or z_1-direction may be calculated by means of expressions

$$dx_i = -u_i^1 dt, \quad i = 2, 3, \quad u_i^1 = u_2^1, u_3^1,$$

(since, in general, the vortex lines approach to the x_1-axis with time, hence, $dy_1 < 0$). The differential equations of a vortex filament trailing behind a lifting surface are (with $dt = (u_1^1)^{-1} dx_1$):

$$x_i - x_{it} = -\int_{x_{1t}}^{x_1} u_i^1 (u_1^1)^{-1} dx_1, \; i = 2, 3, \; u_i^1 = u_2^1, u_3^1, \tag{2.3.3}$$

or

$$x_i - x_{it} = -\int_{x_{1t}}^{x_1} \phi^1_{,x_i} (\varphi^1_{,x_1})^{-1} dx_1, \quad i = 2, 3. \tag{2.3.4}$$

Transforming onto (x, y, z)-or (x_2, y_2, z_2)-system results in:

$$y_2 - y_{2t} =, \text{ or } z_2 - z_{2t} = -\mu^2 \int_{x_{2t}}^{x_2} \varphi_{,i} (U + \varphi_{,x_2}^{\text{II}})^{-1} dx_2, \; i = y_2, z_2. \tag{2.3.5}$$

To find y_{2o}, one has to insert into the left-hand side of (2.3.5) the expression $y_{2o} - s_2$, and on the right-hand side to apply the limits of integration from x_{2t} up to x_2. From equations

$$dx_i = \pm a_i dt, \quad i = 1 \text{ to } 3, \quad a_i = \varphi^1_{,x_i},$$

one gets:

$$a_1^{-1}dx_1 = -a_2^{-1}dy_1 = -a_3^{-1}dz_1,$$

or

$$y_1 = -\int a_1^{-1}a_2 dx_1 + y_{10}(t), \quad z_1 = -\int a_1^{-1}a_3 dx_1 + z_{10}(t), \tag{2.3.6}$$

where the parametric representation of the initial curve (trailing edge) is given by the expression:

$$z_{10}(t) = z_{10}(x_{10}(t), y_0(t)) \quad \text{(Courant and Hilbert, II , p 55).}$$

The parameter t may be eliminated and the function z_1, representing the vorticity surface behind the trailing edge, may be given in the form $z_1 = z_1(x_1, y_1)$. Assume

$$z_{10}(t) = \chi_1(y_{10}(t)),$$

then:

$$z = z_1 = -\int a_1^{-1}a_3 dx_1 + \chi_1(y_1 + \int a_1^{-1}a_2 dx_1), \tag{2.3.7}$$

and in (x, y, z)- or (x_2, y_2, z_2)-system, since $z_1 = \mu^{-1}z_2$, the same surface is given by:

$$z = \mu^{-1}z_2 = -\mu\left[\int (U+\varphi_{,x_2})^{-1}\varphi_{,z_2}dx_2\right] + \chi_2[y_2 + \int (U+\varphi_{,x_2})^{-1}\varphi_{,y_2}dx_2].$$

Let:

$$x_{10}(t) = at+d, \quad y_{10}(t) = bt+e, \quad z_{10}(t) = ct+f,$$

where small letters denote constants. Then,

$$z_{10} = cb^{-1}(y_{10}-e)+f,$$

and the second term on the right-hand side of (2.3.8) transforms into

$$cb^{-1}[\mu^{-1}y_2 + \mu\int (U+\varphi_{,x_2})^{-1}\varphi_{,y_2}dx_2 - e] + f.$$

When the trailing edge is parallel to (x_2, y_2)-plane, then $c \equiv 0$, $z_{10} = f$, and equation simplifies enormously. An element of area of the surface $z_2 = z_2(x_2, y_2)$ is given by (1.4.1) with $dS = dx_2 dy_2$. The partial derivative z_{2,x_2} is equal to

$$(-\mu\varphi_{,z_2} + cb^{-1}\varphi_{,y_2})(U+\varphi_{,x_2})^{-1},$$

and z_{2,y_2} may be obtained by means of differentiation under the integral sign.

PART III. FINAL FORM OF EQUATION.

3.1. Integro-differential Equation.

The problem of the down- and side-wash behind a finite lifting surface may be easily reduced to that one discussed in Part I. The same procedure is applied as in (Heaslet and Lomax, 1948, *a*, *b*; 1947). A point $P(x, y, z)$ located above the lifting surface is assumed. It lies on the surface of the volume of integration which volume is cut by the lifting surface into two parts. Hence, the surface is also cut into two parts, upper and lower, respectively:

$$S_2 = S_{2u} + S_{2l}.$$

S_{2u} consists of the forecone Γ of P, "front surface" λ (envelope of a family of Mach cones having vertices on the leading edge) and the upper area of the lifting surface τ_{2u}. The Mach cones in the (x, y, z) or (x_2, y_2, z_2)-system of coordinates have their axes coinciding with or parallel to the x_2-axis and their vertex angles are 90°. Thus, a conormal lies on λ. The equation of Γ is $r_0^2 = (x-x_2)^2-(p-y_2)^2-(z-z_2)^2 = 0$. Since P lies on S_{2u}, p in (1.2.7) is equal to 2π. Since P does not lie on S_{2l}, $p = 0$ in that case. Subtraction of those two equations of the form (1.2.7) gives the result:

$$2\pi\varphi = -\left\{\iint_{S_{2u}} R_u dS_2 - \iint_{S2l} R_l dS_2\right\}, \quad R_i = r^{-1}\varphi_{i,\nu i} - \varphi_i(r^{-1})_{,\nu i} - a_{i\nu}, \tag{3.1.1}$$

with $i = u, l$. As is known, r^{-1} represents a unit source and $(r^{-1})_{,\nu i}$ a unit doublet. The quantity r^{-1} becomes infinite along Γ. To overcome this difficulty, Hadamard's method of the finite part of an integral is applied (Hadamard, 1913). Regarding the integration over λ, in (Heaslet and Lomax, 1948*a*, p. 30), it was shown that $\varphi = \varphi_{,\nu} = 0$ on λ. Since any derivative $\varphi_{,i}$ may be represented as $\varphi_{,\nu}\nu_{,i}$, it is seen that G(1.1.3), M, a(1.2.6) and a_ν(3.1.1) vanish on λ and the integration over λ is excluded. Thus, only the integration over τ_{2u} and τ_{2l} should be performed. Since the surface of discontinuity consists of the lifting surface and of the vorticity sheet behind the lifting surface, the final form of (3.1.1) is:

$$2\pi\varphi = -\left[\iint_{\tau_2} (R_u - R_l) dS_2 + \iint_{\tau_{2w}} (R_{uw} - R_{lw}) dS_2\right], \tag{3.1.2}$$

where the subscript w denotes the vorticity sheet and R_i's in all the four terms are given by the second form in (3.1.1).

3.2. Boundary Value Problem.

Usually, it may not be possible to satisfy the boundary conditions in an exact, but only in an approximate way. For this reason, it has to be discussed seperately in each particular case. Two kinds of boundary value problems are the most common: (*a*) The load distribution over the lifting surface of zero thickness is supposed to be known and given in the form of $\Delta\varphi_s$ or $(u_u - u_l)$ (2.2.2). Briefly, the jump in the velocity potential over the area of the lifting surface is known. (*b*) The lifting surface is of a finite thickness and shape of the airfoil is given. The slope of a cross-section (airfoil) must be equal to the slope of the streamlines at the surface, i.e.,

$$\xi_{,x_1} = [\varphi_{,z_1}/(U+\varphi_{,x_1})]_{\text{surf.}}, \quad \xi_{,x_2} = \mu[\varphi_{,z_2}/(U+\varphi_{,x_2})]_s, \tag{3.2.1}$$

where $\xi = \xi(z_1, y_1)$, or $\xi = \xi(x_2, y_2)$ denotes the function describing the airfoil section, including the angle of attack. As is known, in the linearized equation $\nabla^2\varphi = 0$ or $\Box^2\varphi = 0$, the function φ may be assumed to represent either acceleration or velocity potential or any of the three perturbation velocity components. The explanation of this fact is simple. Differentiating $\Box^2\varphi = 0$ partially with respect to x_2, y_2 or z_2 and letting $\varphi_{,x_2} = \varphi$, we see that $\Box^2\varphi = 0$ does not change its form. Similarly, from Prandtl's definition of the acceleration potential (Heaslet and Lomax, 1948, p. 15,) ($\zeta_{,x_2} = V_0\varphi_{,x_2x_2}$, $\zeta_{,y_2} = V_0\varphi_{,x_2y_2}$, etc.), we see that $\zeta = \varphi_{,x_2}$ and hence, again $\Box^2\varphi = 0$, may be used to find ζ. In the present case, it is obvious that this procedure is much more complicated. Fundamentally, equation $\Box^2\varphi = G$, refers only to the velocity potential. In case one would like to put $\varphi = \varphi_{,x_2} = u$, the form of the function $G_{,x_2}$ is not identical with that of G. If one performs the indicated differentiation $G_{,x_2}$ and puts $\varphi_{,x_2} \equiv \varphi$, one can use the general method presented in the present work. Of course, on the righthand side of (1.1.3) there will appear new integral forms since, for example, $\varphi_{,y_2}$ must be represented as

$$\int \varphi_{,y_2x_2} dx_2 = \int \varphi_{,y_2} dx_2, \text{ etc.}$$

PART IV. SAMPLE OF APPLICATION.

4.1. Lifting Surface.

For illustrative purposes, the theory presented above will be applied to the simplest possible case of a lifting surface. Assume in (x_1, y_1, z_1)-or (x_2, y_2, z_2)-system a flat lifting surface $\tau_1(\tau_2)$ of finite dimensions, zero thickness, with the leading edge coinciding with, and trailing edge parallel to the y_1-(y_2)-axis. Let in $\{x_1\}$-system the induced velocity component normal to the surface be a continuous function across τ_1, i.e., if subscript s denotes the surface:

$$\Delta\varphi_{s,n_1} = (\varphi_u - \varphi_l)_{,n_1} = \Delta\varphi_{s,x_1}n_{11} + \Delta\varphi_{s,y_1}n_{12} + \Delta\varphi_{s,z_1}n_{13} = 0, \tag{4.1.1}$$

with $n_{1u} = n_{1l} = n_1(n_{11}, n_{12}, n_{13})$. Moreover, $\nu_{1u} = \nu_{1l} = \nu_1(\nu_{11}, \nu_{12}, \nu_{13})$. From the normal form of the equation of a plane, i.e., $x_1 \cos\alpha_1 + y_1 \cos\beta_1 + z_1 \cos\gamma_1 = p_1$, with $\alpha_1 = \pi/2 - \delta_1$, $\beta_1 = \pi/2$, $r_1 = 2\pi - \delta_1$, $p_1 = 0$, and $n_{11} = -\nu_{11} = \sin\delta_1$, $n_{12} = \nu_{12} = 0$, $n_{13} = \nu_{13} = \cos\delta_1$, one easily obtains the equation of the lifting surface

$$z = z_1 = f_1(x_1, y_1) = -x_1 \tan\delta_1,\ F_1(x_1, y_1, z_1) = x_1 \tan\delta_1 + z_1 = 0. \tag{4.1.2}$$

Hence, (4.1.1) may be written in the form

$$\Delta\varphi_{s,z_1} + \Delta_{s,x_1} \tan\delta_1 = 0,$$

where

$$\Delta\varphi_s = \Delta\varphi_s(x_1, y_1, z_1 = -x_1 \tan\delta_1),$$

since the value of φ_s should refer to points on the lifting surface. Since the applied transformation $x_2 = x_1$, $y_2 = \mu y_1$, $z_2 = \mu z_1$, denotes stretching in y-and z-directions, the corresponding surface in the $\{x_2\}$-system is again a flat one passing through the y_2-axis whose equation is

$$z = \mu^{-1}z_2 = f_2(x_2, y_2) = -\mu^{-1}x_2 \tan\delta_2,\ F_2 = x_2 \tan\delta_2 + z_2 = 0. \tag{4.1.3}$$

By comparison, one easily obtains the relation between the angles of attack:

$$\tan\delta_2 = \mu \tan\delta_1,\ \ \sin\delta_2 = \mu B \sin\delta_1,\ \ \cos\delta_2 = B \cos\delta_1, \tag{4.1.4}$$

$$B = [1 + (\mu^2 - 1) \sin^2\delta_1]^{-\frac{1}{2}}, \tag{4.1.4a}$$

where the positive sign of the square root should be taken. With $n_2 = n_2(n_{21}, n_{22}, n_{23})$, $\nu_2 = \nu_2(\nu_{21}, \nu_{22}, \nu_{23})$, $n_{21} = -\nu_{21}$, $n_{22} = \nu_{22}$, $n_{23} = \nu_{23}$, one has

$$\frac{\partial}{\partial \nu_2} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x_2}\frac{\partial x_2}{\partial \nu_2} + \frac{\partial}{\partial y_1}\frac{\partial y_1}{\partial y_2}\frac{\partial y_2}{\partial \nu_2} + \frac{\partial}{\partial z_1}\frac{\partial z_1}{\partial z_2}\frac{\partial z_2}{\partial \nu_2},$$

which in the present case (on the surface), gives:

$$\frac{\partial}{\partial \nu_2} = -\frac{\partial}{\partial x_1}\sin\delta_2 + \frac{\partial}{\partial z_1}\mu^{-1}\cos\delta_2. \qquad (4.1.5)$$

Hence, one may easily derive the formula :

$$\Delta\varphi_{s,\nu_2} = B_1\Delta\varphi_{s,x_1}, \quad B_1 = -(\mu^2+1)\mu^{-1}B\sin\delta_1, \qquad (4.1.6)$$

with $\Delta\varphi_s$ referring to points on the surface. Since

$$r^2 = (x-x_2)^2-(y-y_2)^2-(z-z_2)^2 = (x-x_1)^2-\mu^2(y-y_1)^2-\mu^2(z-z_1)^2,$$

after a few transformations, one has:

$$(r^{-1})_{,\nu_2} = -r^{-3}B\mu[(x-x_1)\sin\delta_1+(z-z_1)\cos\delta_1]. \qquad (4.1.7)$$

The functions a_ν, $\nu = u$ or l, may be represented by means of (1 2.6).

$$a_\nu = \frac{1}{3}\left\{\int_{x_1}^{x} r^{-1}G_\nu dx_1 - \int_{y_1}^{y} r^{-1}G_\nu\mu dy_1 - \int_{z_1}^{z} r^{-1}G_\nu\mu dz_1\right\}, \qquad (4.1.8)$$

$$G_\nu = (c_\infty^2-\beta Y_{1\nu})^{-1}(F_{\nu_1}+F_{\nu_2}), \quad Y_{1\nu} = 2U\varphi_{\nu,x_1}+\textstyle\sum_1^3(\varphi_{\nu,i})^2, \qquad (4.1.8a)$$

$$F_{\nu_1} = \beta M_\infty^2 Y_{1\nu}\varphi_{\nu,x_1x_1}, \qquad (4\ 1.8b)$$

$$F_{\nu_2} = 2U\varphi_{\nu,i}\varphi_{\nu,x_1i}+\varphi_{\nu,i}\varphi_{\nu,k}\varphi_{\nu,ik}. \qquad (4.1.8c)$$

With $z = z_1 = -x_1\tan\delta_1 = \mu^{-1}x_2\tan\delta_2$, $D_1 = \mu$, $D_2 = \mu\tan\delta_1$, $D_3 = 0$, one gets

$$\iint\ldots dS_2 = \iint\ldots\mu(1+\tan^2\delta_1)^{1/2}dx_1dy_1,$$

which may be verified by applying (1.4.1) and (4.1.3) and transforming the so obtained result $[1+\mu^{-2}\tan^2\delta_2]^{1/2}dx_2dy_2$ onto the (x_1, y_1, z_1)-space by means of (4.1.4) and $dy_2 = \mu dy_1$. Thus, the formula for φ over the lifting surface area takes the form:

$$2\pi\varphi = -K\iint_{\tau_1}\{B\mu r^{-3}[(x-x_1)\sin\delta_1+(z-z_1)\cos\delta_1]\Delta\varphi_s + B_1r^{-1}\Delta\varphi_{s,x_1}-(a_u-a_l)\}dx_1dy_1, \qquad (4.1.9)$$

$$K = \mu(1+\tan^2\delta_1)^{1/2}, \quad r^2 = (x-x_1)^2-\mu^2[(y-y_1)^2+(z-z_1)^2], \qquad (4.1.9a)$$

with B given by (4.1.4a), B_1 by (4.1.6) and a_ν by (4.1.8), and $\Delta\varphi_s = \Delta\varphi_s(x_1, y_1, z_1 = -x_1\tan\delta_1)$. The area of the integration τ_1 is the projection on (x_1, y_1)-plane of the area contained inside the curve of intersection of the plane of the lifting surface with the forecone Γ. In $\{x_2\}$-space, the flat surface is given by (4.1.3) and the forecone by $r_0^2 = (x-x_2)^2+\cdots$. Eliminating z_2 by means of (4.1.3) results in

$$(x-x_2)^2-(y-y_2)^2-(z+x_2\tan\delta_2)^2 = 0. \qquad (4.1.10)$$

Equations (4.1.8), i.e., $z_2 = z_2(x_2)$, and (4.1.10), i.e., $y_2 = y_2(x_2)$, represent the curve of intersection in the space $\{x_2\}$. The projection of that curve on the (x_2, z_2)-plane $(y_2 = 0)$ is a straight line, and its projection on the (x_2, y_2)-plane $(z_2 = 0)$ is a hyperbola (4.1.10). In $\{x_1\}$-space that curve may be found in an analogous way or by means of a direct transformation from $\{x_2\}$-onto $\{x_1\}$-space and by use of (4.1.4).

4.2. Vorticity Sheet.

Assume that the vorticity sheet τ_{1w} or τ_{2w}, extending from the trailing edge of the lifting surface $\{x_{1T}\}$ or $\{x_{2T}\}$ up to infinity is a curved surface of zero thickness, that $\Delta p = 0$ and the induced velocity component normal to the vorticity sheet is a continuous function across τ_w. From (2.3.7) with $z_{10} = z_{1T} = f = \text{const.}$, the equation of that surface is:

$$F_1 = z_1 + \int (U + \varphi_{,x_1})^{-1} \varphi_{,z_1} dx_1 - f = 0.$$

The direction cosines of the inward normal to a surface $n_{1i} = F_{1,x_i} K_1$, $K_1 = [\sum_i (F_{1,x_i})^2]^{-1/2}$ can be easily found by means of differentiation under the integral sign in (4.2.1). The condition of a continuous induced normal velocity component (4.1.1) gives

$$\Delta\varphi_{sw,z_1} = -a_1 \Delta\varphi_{sw,x_1} - b_1 \Delta\varphi_{sw,y_1}, \tag{4.2.2}$$

with $-a_1 = -n_{11} n_{13}^{-1} = \nu_{11} \nu_{13}^{-1}$ and $b_1 = n_{12} n_{13}^{-1} = \nu_{12} \nu_{13}^{-1}$, or

$$\Delta\varphi_{sw,\nu_2} = \Delta\varphi_{sw,x_1} (\nu_{21} - \mu^{-1} a_1 \nu_{23}) + \Delta\varphi_{sw,y_1} (\nu_{22} - b_1 \nu_{23}) \mu^{-1}. \tag{4.2.3}$$

The condition $\Delta p = 0$ leads to $Z_{1u} = Z_{1l}$ or $Y_{1u} = Y_{1l}$ on the surface, which gives

$$2U \Delta\varphi_{sw,x_1} + \Sigma_i \Delta\varphi_{sw,x_i} \varphi^{(+)}_{sw,x_i} = 0, \; i = 1 \text{ to } 3, \tag{4.2.4}$$

where $\varphi^{(+)}_{sw,x_i} = (\varphi_u + \varphi_l)_{sw,x_i}$. Insert the value $\Delta\varphi_{sw,z_1}$ from (4.2.2) into (4.2.4), calculate from the resultant equation $\Delta\varphi_{sw,y_1}$ and insert it into (4.2.3), thus obtaining:

$$\Delta\varphi_{sw,\nu_2} = \Delta\varphi_{sw,x_1} [\nu_{21} - \mu^{-1} a_1 \nu_{23} - a_2 (\nu_{22} - b \nu_{23}) \mu^{-1}], \tag{4.2.5}$$

$$a_2 = [2U + \varphi^{(+)}_{sw,x_1} - a_1 \varphi^{(+)}_{sw,z_1}] [\varphi^{(+)}_{sw,y_1} - b_1 \varphi^{(+)}_{sw,z_1}]^{-1} \tag{4.2.5a}$$

The vorticity surface in the $\{x_2\}$-space is given by the formula.

$$F_2 = z_2 + \mu^2 \int (U + \varphi_{,x_2})^{-1} \varphi_{,z_2} dx_2 - \mu f = 0. \tag{4.2.6}$$

By means of partial differentiation one can find

$$F_{2,x_2} = \mu F_{1,x_1}, F_{2,y_2} = F_{1,y_1}, F_{2,z_2} = F_{1,z_1}, K_2 = [\Sigma_i (F_{2,x_i})^2]^{-\frac{1}{2}} = K_1 M, \; x_i = x_2, y_2, z_2,$$
$$M = [1 + (\mu^2 - 1) F_{1,x_1}^2 K_1^2]^{-\frac{1}{2}},$$

which gives the formulas:

$$n_{21} = -\nu_{21} = \cos\alpha_2 = F_{2,x_2} K_2 = \mu M n_{11} = -\mu M \nu_{11}. \tag{4.2.7}$$

$$n_{22} = \nu_{22} = \cos\beta_2 = F_{2,y_2} K_2 = M n_{12} = M \nu_{12}, \tag{4.2.7a}$$

$$n_{23} = \nu_{23} = \cos\gamma_2 = F_{2,z_2} K_2 = M n_{13} = M \nu_{13}. \tag{4.2.7b}$$

For a flat vorticity sheet parallel y_2-axis, the above equations reduce to those derived in section 4.1. Inserting (4.2.7) into (4.2.5) gives

$$\Delta\varphi_{sw,\nu_2}^{\prime\prime} = \Delta\varphi_{sw,x_1}\mu^{-1}\nu_{11}(\mu^2+1)M, \tag{4.2.8}$$

which for a flat vorticity surface transforms into (4.1.6). In the expression $\Delta\varphi_{sw} = \Delta\varphi_{sw}(x_1,y_1,z_1)$, the variable z_1 must be calculated from (4.2.1). Next,

$$(r^{-1})_{,\nu_2} = r^3\mu M[(x-x_1)\nu_{11}-(y-y_1)\nu_{12}-(z-z_1)\nu_{13}], \tag{4.2.9}$$

and $(a_u - a_l)$ is given by (4.1.8). Using the results of previous sections, by means of (4.2.1) and (4.2.6), one easily obtains $D_1 = \mu$, $D_2 = -\mu z_{,x_1}$, $D_3 = -z_{,y_1}$, and

$$\iint\limits_{\tau_{2w}} \ldots dS_2 = \iint\limits_{\tau_{1w}} \ldots [\mu^2+\mu^2(z_{,\text{[illegible]}})^2+z^2_{,y_1}]^{\frac{1}{2}}dx_1dy_1, \tag{4.2.10}$$

where $z_{1,x_1} = -F_{1,x_1}$, $z_{1,y_1} = -F_{1,y_1}$ (4.2.1). To verify this result, use (1.4.1), (4.2.6) with $z_{,x_2} = \mu^{-1}z_{2,x_2} = z_{,x_1}$, $z_{,y_2} = \mu^{-1}z_{2,y_2} = \mu^{-1}z_{,y_1}$, and obtain:

$$\iint\limits_{\tau_{2w}} \ldots dS_2 = \iint\limits_{\tau_{1w}} \ldots [1+(\mu^{-1}z_{2,x_2})^2+(\mu^{-1}z_{2,y_2})^2]^{\frac{1}{2}}dx_2dy_2, \tag{4.2.11}$$

which transforms into (4.2.10),

The area of integration $\tau_{1w}(\tau_{2w})$ is again the area of the projection on $\{x_1\}$-or $\{x_2\}$-plane of the area contained inside the curve of intersection of the voticity surface with forecone Γ belonging to $P(x, y, z)$. In $\{x_2\}$-space the vorticity surface is given by (4.2.6) and the forecone by $r_c^2 = (x-x_2)^2-(y-y_2)^2-(z-z)^2 = 0$. After performing the indicated integration in (4.2.6) one has to calculate the value of z_2 and to insert it into $r_c^2 = \ldots$, to eliminate z_2. That equation gives the desired projection of the curve. In $\{x_1\}$-space a similar procedure must be used. Calculation of z_1 from (4.2.1) and inserting it into $r_c^2 = (x-x_1)^2-\mu^2(y-y_1)^2-\mu^2(z-z_1)^2 = 0$, gives the projection of the desired curve on the (x_1, y_1)-plane. Of course, those curves are not necessarily hyperbolas of the second degree. Hence, one gets as the final result:

$$\begin{aligned}2\pi\varphi_w = -\iint\limits_{\tau_{1w}} \{&-M\mu r^{-3}[(x-x_1)\nu_{11}-(y-y_1)\nu_{12}-(z-z_1)\nu_{13}]\Delta\varphi_{sw} \\ &+M\mu^{-1}(\mu^2+1)\nu_{11}r^{-1}\Delta\varphi_{sw,x_1}-(a_\mu-a_l)\}\cdot[\mu^2+\mu^2(z_{1,x_1})^2+(z_{1,x_1})^2]^{\frac{1}{2}}dx_1dy_1,\end{aligned} \tag{4.2.12}$$

with M given above (4.2 7), K_1 below (4.2.1), F_{1,x_1}, F_{1,y_1}, F_{1,t_1}, by (4.2.1), ν_{1i}, $i =$ 1 to 8, by means of n_{1i}, below (4.2.1), a_ν, $\nu = u$ or l, by (4.1.8), z_{1,x_1} and z_{1,y_1} by (4.2.1). For a flat vorticity surface (4.2.12) transforms onto (4.1.9). By means of (2.3.5) one may obtain the magnitudes of the displacements of the axis of the core:

$$y_{1c} = s_1 - \int_{x_{1TR}}^{x_1} (U+\varphi_{,x_1})^{-1}\varphi_{,y_1}dx_1, \tag{4.2.13}$$

$$z_{1c} = z_{1TR} - \int_{x_{1TR}}^{x_1} (U+\varphi_{,x_1})^{-1}\varphi_{,z_1}dx_1, \tag{4.2.13a}$$

which define in a unique way the shape of the space curve, the axis of the core. The quantity $\Delta\varphi_{sw}$ on the vorticity sheet is equal to $\Gamma_w \equiv \Gamma P$, which quantity is given by the equation $\Gamma_w = \Gamma P$(2.3.1). and was widely discussed in section 2.3.

4.3. Boundary Value Problem for the Lifting Surface.

The quantity $\varphi^{(n)}$ calculated by means of Picard's iteration process, should satisfy the boundary condition, *i.e.*, the solid surface is a stream-surface (consists of streamlines). This condition will be satisfied in an approximate way by requiring that the quantity $\Delta\varphi_s^{(n)}$, to be used on the right-hand side of (4.1.9) to calculate $\varphi^{(n+1)}$ should satisfy the boundary condition on the lifting surface. To calculate $\Delta\varphi_s^{(n)}$, $n \geqq 1$, the formula (2.2.2) will be used with $u_{1\iota}^{(n)} = \varphi_{\iota, x_1}^{(n)}$, $v_{1\iota}^{(n)} = \varphi_{\iota, y_1}^{(n)}$, $w_{1\iota}^{(n)} = \varphi_{\iota, z_1}^{(n)}$, $\iota = u$ or l, and $z_1 = -x_1 \tan \delta_1$ (*i.e.*, $n_1 \rightarrow 0$). The values of $\varphi_{\iota, z_1}^{(n)}$ are defined from the condition that the solid body be a stream-surface, *i.e.*,

$$\varphi_{\iota, z_1}^{(n)} = (U + \varphi_{\iota, x_1}^{(n)}) h_{,x_1}, \tag{4.3.1}$$

with $$\varphi_\iota{}^{(n)} = \varphi_\iota{}^{(n)}(x_1, y_1, z_1 = -x_1 \tan \delta_1),$$

and $h_1 = h_1(x_1, y_1, z_1)$ being the equation of the lifting surface. In the present case, $h_{,x_1} = \tan \delta_1$, and with $dz_1 = -(\tan \delta_1) dx_1$, equation (2. 2. 2) takes the form:

$$\Delta\varphi^{(n)} = (1 - \tan^2 \delta_1) \int_{x1LE}^{x1TR} \Delta\varphi_{s, x_1}^{(n)} dx_1 + \int_{y1TE}^{y1TR} \Delta\varphi_{s, y1}^{(n)} dy_1, \tag{4.3.2}$$

Since $\varphi^{(o)}$ is identically equal to zero, one has to apply at the begining some forms of functions $f^{(u)}$, $f^{(l)}$, discussed in Section 1.3. One may use known results based upon the lineariszed theory and a few particular cases are cited below.

(*a*) *Unswept Wing of Infinite Span.*

For this so-called Ackeret's type of lifting surface

$$\Delta\varphi_s = (U2q)^{-1} \int_{x_{1LE}}^{x_{1TR}} \Delta p dx_1, \text{ with } (\Delta p) q^{-1} = 4\mu^{-1} \tan \delta_1 \approx 4\mu^{-1}\delta_1,$$

so that when the leading edge coincides with the y_1 axis *i. e.*, $\int_0^{x_1} \cdots dx_1$, the circulation per unit span is equal to:

$$\Gamma^{(o)}(x_1) \equiv \Delta\varphi_s{}^{(o)} \equiv 2V\mu^{-1}x_1 \tan \delta_1 \approx 2U\mu^{-1}x_1\delta_1, \tag{4.3.3}$$

and at the trailing edge or in the wake:

$$\Gamma^{(o)}(x_{1T}) \equiv \Delta\varphi_{sw}^{(o)} = 2U\mu^{-1}c_1 \tan \delta_1 \approx 2U\mu^{-1}c_1\delta_1. \tag{4.3.4}$$

where c_1 denotes the uniform chord length.

(*b*) *Tringular Lifting Surface with Leading Edge Swept Back of the Mach Cone from the Vertex.*

$\Delta\varphi_s$ was calculated in (Heaslet, Lomax and Jones, 1947), is cited in (Heaslet and Lomax, 1948*b*):

$$\Gamma^{(o)} \equiv \Delta\varphi_s^{(o)} = 2U\delta_1(E_o\mu)^{-1}(\theta_0^2 x_1^2 - \mu^2 y_1^2)^{\frac{1}{2}} \tag{4.3.5}$$

where E_0 is the complete elliptic integral of the second kind with modulus

$$k_0 = (1-\theta_0^2)^{\frac{1}{2}}, \quad \text{with} \quad \theta_0 = \mu \tan k,$$

k being the semivertical angle of the triangle. At the trailing edge and in the wake the value of $\Delta\varphi_{sw}^{(o)}$ is given by (4.3.5) with $x_1 = c_0$ (root chord). This represents the circulation per unit span at the coordinate y_1

(*c*) *Tringular wing with leading edges swept ahead of the Mach cone from the vertex.*

Equations (4.3.3) and (4.3.4) may be used with c_1 denoting the variable length of the chord.

(*d*) *Rectangular, Trapezoidal, Triangular and some other Planform.*

These are discussed in (Heaslet, Lomax and Jones, 1947).

In case of an unswept wing, the total force perpendicular to the lifting surface is $P = 4\mu^{-1}S_1 q \tan\delta_1$, the lift is $L = P\cos\delta_1$, $c_L = 4\mu^{-1}\sin\delta_1$, $c_D = c_L \tan\delta_1$.

4.4. First Approximation of Picard's Process (lifting surface).

The following assumption are made: $\Delta\varphi_s^{(o)}$ is given by (4.3.3), $a_\nu \equiv 0$, $\nu = u$ or l, the leading edge coincides with the y_1-axis, the area of integration τ_1 is the part of a parabola contained between the following values of the coordinates:

x_1-direction: $x_1 = 0$, and with $y_1 = y + \varepsilon$:

$$x_{1A} = (1-\mu^2\tan^2\delta_1)^{-1}\{(x+\mu^2 z\tan\delta_1) - \mu[\epsilon^2(1-\mu^2\tan^2\delta_1) + (z + x\tan\delta_1)^2]^{\frac{1}{2}}\}; \tag{4.4.1}$$

y_1-direction: $y_1 = y + \epsilon$, and:

$$y_{1A} = y + \mu^{-1}[(x-x_1)^2 - \mu^2(z + x_1\tan\delta_1)^2]^{\frac{1}{2}}. \tag{4.4.2}$$

In case the (x_1, z_1)-plane passes through the point $P(x, y, z)$, one has $y = 0$ in (4.4.2) Since there is a symmetry with respect to the vertical plane passing through P, with (4.1.9) and (4.3.3) one obtains:

$$-\pi(KU)^{-1}\varphi^{(1)} = \mathrm{I}_1 + \mathrm{I}_2, \tag{4.4.3}$$

$$\mathrm{I}_1 = 2B\tan\delta_1(x\sin\delta_1 + z\cos\delta_1)\lim_{\epsilon\to 0}\int_0^{x_{1A}} x_1\,dx_1 \cdot \Big|\overline{\int_{y+\epsilon}^{y_{1A}} r^{-3}dy_1}, \tag{4.4.3a}$$

$$\mathrm{I}_2 = 2B_1\mu^{-1}\tan\delta_1\lim_{\epsilon\to 0}\int_0^{x_{1A}} dx_1\int_{y+\epsilon}^{y_{1A}} r^{-1}dy_1. \tag{4.4.3b}$$

The calculate the finite part of the integral in (4.4.3*a*), introduce the symbol $\mu^2(z + x_1\tan\delta_2)^2 - (x-x_1)^2 = -F_1^2(x_1)$, apply the procedure described in Section 1.5., and by means of elementery methods obtains the indefinite integral:

$$\int (XX^{\frac{1}{2}})^{-1}dy_1 = 2(2cy_1 + b)(qX^{\frac{1}{2}})^{-1} \tag{4.4.4}$$

where $X \equiv r^2 = a + by_1 + cy_1^3$, $a = F_1^2 - \mu^2 y^2$, $b = 2\mu^2 y$, $c = -\mu^2$, $q = 4ac - b^2$, or

$$\mathrm{I}_{1\,indef} = F(x) + c = (y - y_1)(-F_1^2 r)^{-1} + c. \tag{4.4.5}$$

Since the function $A(y_1)$ (1.5.8) is equal to unity, the Taylor's series reduces to this one term and (1.5.4) gives:

$$C = \lim_{y_1 \to y_{1A}} [(-F_1^2 r)^{-1}(y - y_{1A} - y + y_1)] = 0, \tag{4.4.6}$$

as one may verify by means L 'Hospital's rule. Thus, one gets from (1.5.5) with $y_1 = y + \epsilon$ as the lower limit:

$$I_{11}^{\bullet} = \int_{y+\epsilon}^{*} r^{-3} dy_1 = -\epsilon F_1^{-2}[(x - x_1)^2 - \mu^2\epsilon^2 - \mu^2(z + x_1 \tan \delta_1)^2]^{-\frac{1}{2}}. \tag{4.4.7}$$

The integral $\int_{y+\epsilon}^{y_{1A}} r^{-3} dy_1$ may be calculated by means of elementary methods since it is finite at both limits. Hence

$$\int_{y+\epsilon}^{y_{1A}} X^{-\frac{1}{2}} dy_1 = -(-c)^{-\frac{1}{2}} \sin^{-1}[(2cy_1 + b)(-q)^{-\frac{1}{2}}]_{y+\epsilon}^{y_{1A}} = \mu^{-1}\left[\frac{\pi}{2} - \sin^{-1}(\mu c F_1^{-1})\right]. \tag{4.4.8}$$

Inserting (4.4.7) and (4.4.8) into (4.4.8), one obtains the formulae:

$$\varphi^{(1)} = M_1 \Pi_1 + M_2 \Pi_2, \tag{4.4.9}$$

$$M_1 = 2U\mu(1 + \tan^2\delta_1)^{\frac{1}{2}}[x \sin^2\delta_1(\cos \delta_1)^{-1} + x \sin \delta_1] M_{11}^{-1}, \tag{4.4.9a}$$

$$M_{11} = \pi[1 + (\mu^2 - 1) \sin^2\delta_1]^{\frac{1}{2}}, \tag{4.4.9b}$$

$$M_2 = 2U(\mu^2 + 1)(1 + \tan^2\delta_1)^{\frac{1}{2}} \sin^2 \delta_1(\cos \delta_1)^{-1}(\mu^2 M_{11})^{-1}, \tag{4.4.9c}$$

$$\Pi_1 = \lim_{\epsilon \to 0} \epsilon \int_0^{x_{1A}} x_1 F_1^{-3} G_1^{-1} dx_1, \tag{4.4.9d}$$

$$\Pi_2 = \lim_{\epsilon \to 0} \int_0^{x_{1A}} \left[\frac{\pi}{2} - \sin^{-1}(\mu\epsilon F_1^{-1})\right] dx_1, \tag{4.4.9e}$$

where

$$G_1^2 = (x - x_1)^2 - \mu^2\epsilon^2 - \mu^2(z + x_1 \tan \delta_1)^2. \tag{4.4.9f}$$

For δ_1 small, $\sin^2\delta_1 \approx \tan^2\delta_1 \approx 0$, $\sin \delta_1 \approx \delta_1$, and (4.4.9a) and (4.4.9c) transform into the expressions $M_1 = 2U\mu\delta_1 z/\pi$, $M_2 = 0$, which are known from the linearized theory (Heaslet and Lomax, 1948*b*, p. 16, eq. (16)). To calculate Π_1, decompose F_1^2 into:

$$F_1^2 = (1 - \mu^2 \tan^2\delta_1)[x_1 - (x - \mu z)(1 + \mu \tan \delta_1)^{-1}]\ [x - (x + \mu z)(1 - \mu \tan \delta_1)^{-}]. \tag{4.4.10}$$

Applying the formula $\int x_1(x_1 - a')^{-1}(x_1 - b')^{-} G^{-1} dx_1$, one obtains $\Pi_1 = \lim_{\epsilon \to 0} \epsilon \cdot (I_{11} - I_{12})$,

where I_{11}, I_{12}, are solved by means of formulas $\int (vG)^{-1} dx_1$, and $\sin^{-1}x - \sin^{-1}y$. With $\epsilon \to 0$, one gets $\Pi_1 = III_1 - III_2$, and

$$III_i = A_{1i} \sin^{-1}[C_{1i}(1 - D_{1i}^2)^{\frac{1}{2}} - D_{1i}(1 - C_{1i}^2)^{\frac{1}{2}}]. \tag{4.4.11}$$

$$A_{1i} = \mp (x \mp \mu z)[2\mu^2(z + x \tan \delta_1)(1 \pm \mu \tan \delta_1)]^{-1}, \tag{4.4.11a}$$

with $C_{1i} = D_{1i} = -1$, $i = 1, 2$. The upper signs refer to $i = 1$, and the lower ones to

$i = 2$. The value $\sin^{-1}0 = \pi$ was chosen since the value 0 gives $\Pi_1 = 0$. The final result is:

$$M_1\Pi_1 = -N_1[(x-\mu z)(1+\mu\tan\delta_1)^{-1}+(x+\mu z)(1-\mu\tan\delta_1)^{-1}], \qquad (4.4.12)$$

$$F_1 = U(1+\tan^2\delta_1)^{\frac{1}{2}}\sin\delta_1\mu^{-1}[1+(\mu^2-1)\sin^2\delta_1]^{-\frac{1}{2}}, \qquad (4.4.12a)$$

For δ_1 small, $\sin^2\delta_1 \approx \tan^2\delta_1 \approx \tan\delta_1 \approx 0$, $\sin\delta_1 \approx \delta_1$, (4.4.12) transform into $M_1\Pi_1 = -U\delta_1\mu^{-1}[(x+\mu z)+(x-\mu z)]$, which is the same as (18) in (Heaslet and Lomax, 1948*b*) If in (4.4.11) $\sin^{-1}0 = 0$, then $M_1\Pi_1 = -N_1(x+\mu z)\cdot(1-\mu\tan\delta_1)^{-1}$, and for δ_1 small, $M_1\Pi_1 = -U\delta_1\mu^{-1}(x+\mu z)$ (linearized theory). This gives $w^{(1)} = \varphi^{(1)}_{,z} = -U\delta_1$, which is the same as (18) in (Heaslet and Lomax, 1948*b*). The second integral (4.4.9*e*) is $\Pi_2 = \mathrm{I}_{21} - \mathrm{I}_{22}$,

$$\mathrm{I}_{21} = \lim_{\epsilon\to 0}\int_0^{x_{1A}} \tfrac{1}{2}\pi\, dx_1 = \tfrac{1}{2}\pi(x-\mu z)(1+\mu\tan\delta_1)^{-1}, \qquad (4.4.13a)$$

$$\mathrm{I}_{22} = \lim_{\epsilon\to 0}\int_0^{x_{1A}} \sin^{-1}(\mu\epsilon F_1{}^{-1})dx_1 = 0. \qquad (4.4.13b)$$

To prove the last result, expand $\sin^{-1}x = x - x^3/6 + \cdots$ Since F_1 does not contain ϵ and the integral itself is bounded for all $0 \leqq \epsilon < \infty$, for $\epsilon \to 0$, $\mathrm{I}_{22} = 0$. Thus $M_2\Pi_2 = N_{21}N_{22}{}^{-1}$,

$$F_{21} = U(\mu^2+1)(1+\tan^2\delta_1)^{\frac{1}{2}}\sin^2\delta_1(\cos\delta_1)^{-1}(x-\mu z), \qquad (4.4.14a)$$

$$N_{22} = \mu^2[1+(\mu^2-1)\sin^2\delta_1]^{\frac{1}{2}}(1+\mu\tan\delta_1). \qquad (4.4.14b)$$

Hence, $\varphi^{(1)} = M_1\Pi_1 + N_{21}N_{22}{}^{-1}$, with $M_1\Pi_1$ given by (4.4.12) and N_{21}, N_{22} by (4.4.14).

4.5. First Approximation of Picard's Process (vorticity sheet).

In the initial approximation one cannot calculate the side-or-downwash since $\varphi^{(0)}_w \equiv 0$. The value of $\Gamma^{(0)}_w$ is given by $\Gamma^{(0)}_w = \Gamma P$, $P = 1$, *i.e.*, by (4.4.4). Hence, in the first approximation the vorticity surface is a flat surface. The area of integration is a parabola consisting of two parts: one on the lifting surface and another on the flat vorticity surface. Assuming that the region of integration is not affected by the Mach cones from the tips of leading or trailing edge (where the situation is different), in the part on the lifting surface $\Delta\varphi^{(0)}_s$ is given by (4.3.3.) and $\varphi^{(1)}_{1w}$ by (4.4.3) with integrals $\int_0^{c_1} x_1 dx_1$ in both (4.4.3*a*) and (4.4.3*b*), where c_1 denotes the chord. Following (Heaslet and Lomax, 1948*b*) write the integral

$$\int_0^{c_1} dx_1 \int_{y+\epsilon}^{y_{1A}} r^{-3}\,\Delta\varphi^{(0)}_{sw} dy_1$$

in the form

$$\int dx_1 \left[\int_{y_1\ldots}^{y_{1A}} - \int_{y_1\ldots}^{y_1+\epsilon}\right].$$

But since the quantity $\Delta\varphi^{(0)}_{sw} r^{-3} \leqq M$ is a bounded quantity for $c_1 \leqq x_1 \leqq x_0$, where x_0 is a certain value however large, then changing the order of integration we may write

$$\lim_{\epsilon\to 0} M\int_{y_1}^{y_1+\epsilon} dy_1 = \lim_{\epsilon\to 0} M\varepsilon = 0.$$

Therefore, for P situated behind the trailing edge wave (envelope of trailing edge Mach cones) the contribution of the region lying on the lifting surface is given by the integrals

$$\int_0^{c_1} x_1 dx_1 \left| \overline{\int_{y_1}^{y_{1A}} r^{-3} dy_1} \right. ; \quad \int_0^{c_1} dx_1 \int_{y_1}^{y_{1A}} r^{-1} dy_1. \tag{4.5.1}$$

The application of Hadamard's procedure shows that the first integral (4.5.1) is equal to zero. The second integral is equal to $\frac{1}{2}\mu^{-1}\pi c_1$ and in the result

$$\varphi_{1w}^{(1)} = U(\mu^2+1)(1+\tan^2\delta_1)^{1/2}\sin^2\delta_1\,(\cos\delta_1)^{-1}c_1 Z^{-1}, \tag{4.5.2}$$

$$Z = \mu\,[1+(\mu^2-1)\sin^2\delta_1]^{1/2}. \tag{4.5.2a}$$

The expression (4.5.2) vanishes for $\delta_1 \equiv 0$ (linearized theory). To calculate the contribution of the part lying on the (flat) vorticity sheet, equation (4.1.9) or (4.4.3) will be used with $\Delta\varphi_{sw}^{(0)} = 2U\mu^{-1}c_1 \tan\delta_1$, $\Delta\varphi_{sw,x_1}^{(0)} = 0$, (hence, $I_2 = 0$). The result is:

$$-\pi(KU)^{-1}\varphi_{2w}^{(1)} = 2B\tan\delta_1[x\sin\delta_1 + z\cos\delta_1]\cdot$$
$$\cdot\lim_{\epsilon\to 0} c_1 \int_{y_1}^{x_{1B}} dx_1 \left| \overline{\int_{y_1+\epsilon}^{y_{1A}} r^{-3} dy_1} \right. . \tag{4.5.3}$$

The finite value of the integral was calculated in Section 4.4 Inserting (4.4.7) into (4.5.3) and applying elementary formulas (see above eq. 4.4.10), after some transformations gives the result:

$$\varphi_{2w}^{(1)} = -2(x\sin\delta_1 + z\cos\delta_1)c_1 U\tan\delta_1 H^{-1}, \tag{4.5.4}$$

$$H = \mu\cos\delta_1[1+(\mu^2-1)\sin^2\delta_1]^{1/2}(z + x\tan\delta_1). \tag{4.5.4a}$$

Higher approximations can be calculated in a similar way.

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CALCUTTA MATHEMATICAL SOCIETY.

Report of the Council for the year 1952 to the Annual General Meeting of the Society

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1952 as required by the provisions of Rule 25.

The Council: The Council of the Society for the year 1952 consisting of the officers and other members elected at the last Annual General Meeting and co-opted thereafter, together with the Editorial Secretary, was constituted as follows:

President

Professor H. D. Bagchi.

Vice-Presidents

Prof. N. R. Sen	Dr. T. Vijayaraghavan
Dr. R. N. Sen	Prof. A. C. Banerjee

Dr. S. K. Chakraborty

Treasurer

Mr. S. C. Ghosh

Secretary

Mr. U. R. Burman

Editorial Secretary

Mr. P. K. Ghosh

Other Members of the Council

Dr. G. Prasad	Prof. S. K. Basu
Prof. B. R. Seth	Dr. S. Ghosh
Mr. B. C. Chatterjee	Dr. B. S. Ray
Prof. B. B. Sen	Dr. N. L. Ghosh
Dr. N. G. Shabde	Prof. N. M. Basu
Dr. A. C. Choudhury	Dr. Ram Behari

Dr. S. S Bhatnagar (Co-opted, representative of the Government of India).

General: The various activities of the Society have been carried on throughout the year in much the same form as in the past few years.

Membership: The Council records its profound sense of sorrow and loss at the death of Prof. Elie Cartan of France, who was an honorary member of the Society and an outstanding mathematician of the present century.

The Council also reports that there have been some additions to the list of members of the Society, eight ordinary and one life-member being enrolled during the year.

Meetings during 1952: The council held six meetings during the year and there were five ordinary general meetings devoted to the reading of original papers communicated to the Society for publication in its Bulletin.

Publications: During the year under review the Society has published four numbers of the Bulletin namely, vol. 43, Nos. 2-4 and vol. 44, No. 1. The Council regrets very much that on account of circumstances beyond its control the publications of the Society are being very considerably delayed and desires to report that with a view to expedite the publications the Society has found it necessary to appoint a trained compositor with effect from July last on a salary of Rs. 75/- per month. This compositor is now engaged at the University Press exclusively in the work of printing the Society's Bulletin and it is expected that if funds be available to enable the Society to maintain this arrangement, the arrears in publications will be wiped off before the current year is out. The Council takes the opportunity to record here the Society's indebtedness to the authorities of the Calcutta University for printing the Bulletin free of charge and the Officers and members of the staff of the Calcutta University Press for their valued services.

Exchange of Publications: The transmission of the Society's publications to various countries in the world has been carried on regularly during the year and some new exchange relations have also been established.

Library: It is gratifying to note that there has been no break during the year under review in the run of periodicals which the Society has been subscribing for the last few years, out of grants received from the Central and State Governments. These grants have also enabled the Society to add to its library a collection of some recent standard books on various branches of Pure and Applied Mathemetics.

The Council desires to report here that a book case has been constructed for the library at a cost of Rs. 500/- to provide space for the current issues of the periodicals, prior to their being sent out to the bindery.

Finance: The annual accounts of the Society for the year under review have been presented to the Council in the standardised form by the auditors Dr. H. M. Sen Gupta and Mr. M. C. Chaki. The Council gratefully acknowledges its indebtedness to them for their honorary services.

The Society received the following grants during the year under review:—

(i).	Government of India	Rs. 1,000/-
(ii).	Government of W. Bengal ...	Rs. 1,000/-
(iii).	Nat. Inst. of Sciences of India ...	Rs. 400/-

and the Council takes this opportunity to offer its grateful thanks to the Central and State Governments and to the National Institute of Sciences for these grants.

CALCUTTA MATHEMETICAL SOCIETY

RECEIPT AND DISBURSEMENTS ACCOUNTS OF THE CALCUTTA MATHEMETICAL SOCIETY FOR THE YEAR ENDING 31ST DECEMBER, 1952.

Receipts

			Rs. As. P
1. Opening Balance			
(a) With Secretary			
(i) In cash ..	0 1 9		
(ii) In stamps ...	5 0 0		
		5 1 9	
(b) Balance at Banks			
(i) Imperial Bank of India	3,419 2 9		
(ii) Do (K. K. G. P. Fund)	576 1 8		
(iii) United Bank of India	2,210 1 3		
(iv) Postal Savings Bank	821 13 0		
		7,027 2 8	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000) ...		5,663 11 6	
(d) G. P. Notes (K. K. G. P. Fund) (Face value Rs. 2,000) ...		1,937 7 9	
			14,633 7 8
2. Membership subscription	964 4 0		
Do. (in suspense)	12 0 0		
3. Admission fees	70 0 0		
4 Sale proceeds	3,301 8 3		
		4,347 12 3	
5. Grants			
(i) Govt. of India	1,000 0 0		
(ii) Govt. of West Bengal	1,000 0 0		
(iii) Nat. Inst. of Sciences	400 0 0		
		2,400 0 0	
6 Interest			
(a) G. P. Notes (General Fund)	180 0 0		
(b) G.P Notes (K.K.G.P. Fund)	60 0 0		
(c) Postal Savings Bank	12 0 0		
		252 0 0	
			6,999 12 3
		TOTAL Rs.	21,633 3 11

Disbursements

			Rs. As. P.
1 Establishment ...		955 0 0	
2. Construction of Book Case ...		500 0 0	
3 Meetings ...		57 12 6	
4. Books & Journals (including binding) ...		3,077 11 0	
5. Bulletins			
(a) Compositor's salary	375 0 0		
(b) Papers, Blocks, Types etc.	1,021 3 0		
(c) Postage	339 5 9		
		1,735 8 9	
6. Printing & Stationery ...		280 5 9	
7. Postage (General) ...		101 3 0	
8. Bank charges ..		18 0 0	
9 Miscellaneous (including conveyance charges)		65 11 6	
			6,791 4
CLOSING BALANCE			
(a) With Secretary			
(i) In cash	12 4 3		
(ii) In stamps	4 7 3		
		16 11 6	
(b) Balance at Banks			
(i) Imperial Bank of India	4,246 14 3		
(ii) Do (K. K. G P Fund)	635 9 8		
(iii) United Bank of India	1,436 11 9		
Do (in suspense)	12 0 0		
(iv) Postal Savings Bank	893 13 0		
		7,224 0 8	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000)		5,663 11 6	
(d) G. P Notes (K K. G. P. Fund) (Face value Rs 2,000)		1,937 7 9	
			14,841 15 5
		TOTAL ... Rs.	21,633 3 11

To

The MEMBERS of the Calcutta Mathematical Society,

We have examined the above Balance Sheet with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

M. C. CHAKI
H. M. SEN GUPTA *Auditors.*

VARIATION OF TEMPERATURE DUE TO SMALL STEADY DISTURBANCES IN A COMPRESSIBLE FLOW

By

M. Ray, *Agra*

(*Received—April 7, 1952*)

Summary

The problem of temperature distribution due to small steady disturbances in a uniform stream in two-dimensions has been discussed. The fluid has been taken slightly compressible, the effects of viscosity and conductivity have been taken into account and the condition has been taken as non-adiabatic. Usual method of neglecting squares and products of the components of disturbed velocity has been adopted. It has been found that variation of temperature is proportional to that of velocity and that the disturbance depends only on one parameter, that is, Prandtl number.

1. **Introduction:** An infinite mass of fluid has a uniform temperature and flows with uniform velocity U in a given direction. In such a fluid small steady disturbances are imagined to propagate, which give rise to small variations of velocity and temperature. Taking into account the effects of viscosity and conductivity, the disturbed velocity will be rotational in character but if this disturbed velocity is small, the equations of motion can be simplified and under some plausible assumptions, can be solved. Temperature distribution is given by the Energy Equation and the above assumptions arise when we go to solve the Energy Equation.

2. **Disturbance Equations:** The direction of the uniform undisturbed stream is taken as x-axis and the flow is supposed to be non-adiabatic and two-dimensional. If the disturbance is small and the fluid is very slightly compressible, then the variations of velocity, temperature and pressure will be all very small. We take pressure as constant. Let $(U+u,\, v)$ be the components of disturbed velocity and if ϱ_0 be the density in the undisturbed state, let $(\varrho_0+\varrho)$ be the density when disturbed. The quantities u, v, ϱ are all small so that their squares and products can be neglected as well as the squares and products of their first partial derivatives.

The equation of continuity is

$$\frac{\partial}{\partial x}\{(\varrho_0+\varrho)(U+u)\}+\frac{\partial}{\partial y}\{(\varrho_0+\varrho)v\} = 0,$$

which simplifies into, with the above assumptions,

$$\varrho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+U\frac{\partial \varrho}{\partial x} = 0. \tag{1}$$

The equations of motion are

$$(\varrho_0+\varrho)\left\{(U+u)\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right\} = \frac{\partial}{\partial x}p_{xx}+\frac{\partial}{\partial y}p_{xy}\,,$$

CALCUTTA UNIVERSITY CENTRAL LIBRARY

and

$$(\varrho_0+\varrho)\left\{(U+u)\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right\}=\frac{\partial}{\partial x}p_{xy}+\frac{\partial}{\partial y}p_{yy},$$

where

$$p_{xx}=2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right),$$

$$p_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$

and

$$p_{yy}=2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right).$$

With the above approximations, these equations simplify into

$$\varrho_0 U\frac{\partial u}{\partial x}=\frac{4}{3}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{2}{3}\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right),\tag{2}$$

and

$$\varrho_0 U\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{4}{3}\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)-\frac{2}{3}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial y}\right).\tag{3}$$

3. Energy Equation: If I be the enthalpy or heat content per unit mass in the undisturbed stream and $(I+i)$ is the enthalpy in the disturbed state, the equation of energy is

$$(\varrho_0+\varrho)\left\{(U+u)\frac{\partial i}{\partial x}+v\frac{\partial i}{\partial y}\right\}=\frac{\partial}{\partial x}\left(\frac{\mu}{\sigma}\frac{\partial i}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{\sigma}\frac{\partial i}{\partial y}\right),\tag{4}$$

where $\sigma=\mu c_p/\kappa$, is the Prandtl number, assumed constant; κ, the thermal conductivity and c_p the specific heat at constant pressure, assumed constant.

It is to be noted that in the right hand side of equation (4) the dissipation function has been neglected in the present approximation as it consists of terms which are squares and products of partial derivatives, so that no heat is generated by dissipation for such a small disturbance.

The equation (4) simplifies into

$$\sigma\varrho_0 U\frac{\partial i}{\partial x}=\frac{\partial}{\partial x}\left(\mu\frac{\partial i}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial i}{\partial y}\right).\tag{5}$$

Here again i and its partial derivatives have been taken as small.

4. Solution of the Equation: If we take $v=0$, the equations (1), (2) and (3) become

$$\varrho_0\frac{\partial u}{\partial x}+U\frac{\partial\varrho}{\partial x}=0,\tag{6}$$

$$\varrho_0 U\frac{\partial u}{\partial x}=\frac{4}{3}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),\tag{7}$$

and

$$0=\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{2}{3}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial x}\right).\tag{8}$$

The equation (8) is satisfied if we put

$$\mu\frac{\partial u}{\partial x} = \frac{\partial\psi}{\partial x} \quad\text{and}\quad \mu\frac{\partial u}{\partial y} = \frac{2}{3}\frac{\partial\psi}{\partial y}, \tag{9}$$

then the equation (7) gives

$$\varrho_0 U\frac{\partial u}{\partial x} = \frac{4}{3}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial x}+\frac{2}{3}\frac{\partial}{\partial y}\frac{\partial\psi}{\partial y}. \tag{10}$$

Now assuming i to be a function of u alone, the energy equation (5) gives

$$\sigma\varrho_0 U\frac{di}{du}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\mu\frac{di}{du}\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial y}\frac{\partial}{\partial u}\left(\mu\frac{di}{du}\frac{\partial u}{\partial y}\right),$$

and therefore from (9) we get

$$\sigma\varrho_0 U\frac{\partial\psi}{\partial x}\frac{di}{du} = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial\psi}{\partial x}\frac{di}{du}\right)+\frac{4}{9}\frac{\partial\psi}{\partial y}\frac{\partial}{\partial u}\left(\frac{\partial\psi}{\partial y}\frac{di}{du}\right). \tag{11}$$

To solve the equation, we further assume that $\partial\psi/\partial x$ and $\partial\psi/\partial y$ are functions of u alone

Then the equation (10) becomes

$$\varrho_0 U\frac{\partial u}{\partial x} = \frac{4}{3}\frac{\partial u}{\partial x}\frac{d}{du}\frac{\partial\psi}{\partial x}+\frac{2}{3}\frac{\partial u}{\partial y}\frac{d}{du}\frac{\partial\psi}{\partial y}$$

or from (9)

$$\varrho_0 U\frac{\partial\psi}{\partial x} = \frac{4}{3}\frac{\partial\psi}{\partial x}\frac{d}{du}\left(\frac{\partial\psi}{\partial x}\right)+\frac{4}{9}\frac{\partial\psi}{\partial y}\frac{d}{du}\left(\frac{\partial\psi}{\partial y}\right)$$

$$= \frac{2}{3}\frac{d}{du}\left(\frac{\partial\psi}{\partial x}\right)^2+\frac{2}{9}\frac{d}{du}\left(\frac{\partial\psi}{\partial y}\right)^2$$

$$= \frac{2}{9}\frac{d}{du}\left\{3\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\}.$$

Therefore

$$\frac{d}{du}\left\{3\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\} = \frac{9}{2}\varrho_0 U\frac{\partial\psi}{\partial x}. \tag{12}$$

Now to solve equations (11) and (12), we must make some plausible assumptions regarding $\partial\psi/\partial x$ and $\partial\psi/\partial y$. Since these are functions of u alone, the simplest and most suitable assumptions are to put,

$$\frac{\partial\psi}{\partial x} = au, \quad \frac{\partial\psi}{\partial y} = bu, \tag{13}$$

where a, b are constants.

Using (13) we get from (12)

$$12a^2+4b^2 = 9\varrho_0 Ua. \tag{14}$$

Again using (13) we get from (11)

$$\sigma\varrho_0 Ua\frac{di}{du} = \left(a^2+\tfrac{4}{9}b^2\right)\left(u\frac{d^2i}{du^2}+\frac{di}{du}\right),$$

or

$$u\frac{d^2i}{du^2}+(1-\alpha)\frac{di}{du} = 0, \tag{15}$$

where

$$\alpha = \sigma\varrho_0 Ua/(a^2+\tfrac{4}{9}b^2). \tag{16}$$

The solution of (15) is

$$i = Cu^{\alpha}/\alpha, \tag{17}$$

taking $i = 0$ when $u = 0$, so that $\alpha > 0$.

Now for a perfect gas we have,

$$i\varrho = \gamma p/(\gamma-1),$$

where $\gamma = c_p/c_v$, c_v being the specific heat at constant volume. This law in the present case gives, since p is constant,

$$(I+i)(\varrho_0+\varrho) = \text{constant} = I\varrho_0$$

or neglecting $i\varrho$, we get

$$\varrho_0 i+I\varrho = 0,$$

therefore

$$\varrho_0\frac{\partial i}{\partial x}+I\frac{\partial \varrho}{\partial x} = 0. \tag{18}$$

Hence with the help of equation (6) we get

$$U\frac{\partial i}{\partial x} = I\frac{\partial u}{\partial x}.$$

Since i is a function of u only, this gives,

$$U\frac{di}{du} = I.$$

Therefore from (17) we get

$$UCu^{\alpha-1} = I.$$

This can be true only if $\alpha = 1$, so that

$$C = I/U$$

and hence from (17), we get

$$i/I = u/U. \tag{19}$$

Now since $\alpha = 1$, we have from (16),

$$9a^2+4b^2 = 9\sigma\varrho_0 Ua. \tag{20}$$

We are now in a position to find the values of the constants a and b.

From (14) and (20) we get,

$$\left.\begin{aligned} a &= 3\varrho_0 U(1-\sigma), \\ \text{and} \qquad & \\ b &= \frac{3\sqrt{3}}{2}\varrho_0 U\sqrt{\{(1-\sigma)(4\sigma-3)\}}. \end{aligned}\right\} \tag{21}$$

Hence we must have $\sigma < 1$ but $> \frac{3}{4}$, which is true practically in all cases

Turning back to the equations (9) and (13) we get

$$\mu\frac{\partial u}{\partial x} = au \quad\text{and}\quad \mu\frac{\partial u}{\partial y} = \frac{2}{3}bu$$

so that

$$\frac{\partial u}{\partial x} = \lambda\frac{\partial u}{\partial y} \tag{22}$$

where

$$\lambda = 3a/2b.$$

Using the values of a and b from (21), we get

$$\lambda = \{3(1-\sigma)/(4\sigma-3)\}^{\frac{1}{2}}. \tag{23}$$

Now the solution of equation (22) is

$$u = f(y+\lambda x). \tag{24}$$

This gives the disturbance.

Thus if a disturbance represented by equation (24) is propagated throughout a uniform steady stream, the variation of temperature will be given by equation (19).

For instance, if we take the disturbance as

$$u = u_0 \sin(y+\lambda x), \tag{25}$$

the lines of no disturbance will be $y+\lambda x = n\pi$ where n is zero or integer and the lines of greatest disturbance will be $y+\lambda x = \frac{1}{2}n\pi$, n being odd integer. On the first set of lines temperature will remain constant whereas on the second set, variations of temperature will be greatest.

5. **Result:** When a small steady disturbance is propagated into a steady uniform flow of a slightly compressible fluid, there will be a small variation of velocity and temperature. As a first approximation pressure will remain constant and there will be no generation of heat by dissipation and variation of temperature will be proportional to that of velocity. In particular we can take these variations to be maximum or minimum along parallel lines cutting the direction of the uniform stream at an angle, the slope depending upon Prandtl number.

AGRA COLLEGE,
AGRA, INDIA

TWO-DIMENSIONAL STEADY FLOWS SUPERPOSABLE ON A SOURCE, SINK OR DOUBLET

By

Prem Prakash, *Lucknow*.

(Communicated by Prof. N R. Sen. Received—April 7, 1952)

Introduction.—The present paper aims at investigating the general type of steady flows that can be superimposed on a two dimensional source, sink or doublet. It is seen that only an irrotational motion can be superimposed on a flow due to either of these; superposability being defined as earlier (Ram Ballabh, 1940).

1. First we take the case of a source or sink. Let a two dimensional source of strength m be situated at the origin. The velocity potential φ is given by

$$\varphi = -m \log r = -\frac{m}{2}\log(x^2+y^2)$$

If (u_1, v_1) be the velocity, we have

$$u_1 = \frac{mx}{x^2+y^2}, \quad v_1 = \frac{my}{x^2+y^2}.$$

Let (u, v) and ζ be the velocity and vorticity for a two dimensional rotational flow superposable on (u_1, v_1). The condition of superposability [Ram Ballabh, *l c*, (1. 3)] gives

$$\frac{\partial}{\partial x}\left(\frac{mx}{x^2+y^2}\cdot\zeta\right)+\frac{\partial}{\partial y}\left(\frac{my}{x^2+y^2}\cdot\zeta\right) = 0$$

which gives

$$x\frac{\partial\zeta}{\partial x}+y\frac{\partial\zeta}{\partial y} = 0 \tag{1.1}$$

of which the general solution is

$$\zeta = f(y/x), \tag{1.2}$$

'f' being an arbitrary function.

The condition of integrability of the equations of rotational motion, assumed steady, is (Lamb, 1945)

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}-\nu\nabla^2\right)\zeta = 0,$$

ν being the kinematic viscosity.

Substituting for ζ from (1.2),

$$f'(y/x)(x^3v-x^2yu) = \nu\{(x^2+y^2)\cdot f''(y/x)+2xy\cdot f'(y/x)\},$$

Putting
$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x},$$

$$x^3\frac{\partial\psi}{\partial x} + x^2 y\frac{\partial\psi}{\partial y} = \nu\left\{\frac{f''}{f'}(x^2+y^2)+2xy\right\}.$$

The auxiliary equations to solve this are

$$\frac{dx}{x^3} = \frac{dy}{x^2 y} = \frac{d\psi}{\nu\left\{\frac{f''}{f'}(x^2+y^2)+2xy\right\}},$$

two solutions of which are

$$y/x = \text{constant, and}$$

$$\psi - \nu\left\{(1+y^2/x^2)\frac{f''}{f'}+2y/x\right\}\log x = \text{constant.}$$

Hence the complete solution is given by

$$\psi = \nu\left\{(1+y^2/x^2)\frac{f''(y/x)}{f'(y/x)}+2y/x\right\}\log x + \chi(y/x), \tag{1.8}$$

where χ is another arbitrary function.

The value of ψ as obtained in (1.8) has to satisfy the requirement

$$\nabla^2\psi = \zeta = f(y/x)$$

i.e.,
$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = f(\tan\theta)$$

transformed into polar coordinates.

Substituting for ψ from (1.3) and writing F for

$$\left\{(1+\tan^2\theta)\frac{f''(\tan\theta)}{f'(\tan\theta)}+2\tan\theta\right\}$$

this becomes

$$\frac{\nu\log(r\cos\theta)}{r^2}\left[F''\sec^4\theta+2F'\sec^2\theta\tan\theta\right] + \frac{1}{r^2}\left[\chi''\sec^4\theta+2\chi'\sec^2\theta\tan\theta\right.$$
$$\left.-\nu F\sec^2\theta-2\nu F'\sec^2\theta\tan\theta\right] = f(\tan\theta).$$

Since there is no r on the right hand side, the left hand side must be a function of θ alone which is possible only if it vanishes identically, *i.e.*, if $f(\tan\theta) = 0$, or the motion $(u, v, \zeta,)$ is irrotational.

2. We next take the case of a two dimensional doublet.

Let a doublet of strength μ be situated at the origin, its axis coinciding with the axis of x. The velocity potential φ is given by

$$\varphi = \frac{\mu\cos\theta}{r} = \frac{\mu x}{x^2+y^2}$$

The velocity at any point of the plane due to the doublet is given by

$$u_1 = \frac{\mu(x^2-y^2)}{(x^2+y^2)^2}, \quad v_1 = \frac{2\mu xy}{(x^2+y^2)^2}.$$

If u, v, be the velocity components and ζ the vorticity of the flow which can be superimposed upon $(u_1, v_1,)$, the condition of superposability gives

$$\frac{\partial}{\partial x}\left\{\frac{\mu(x^2-y^2)}{(x^2+y^2)^2}\zeta\right\} + \frac{\partial}{\partial y}\left\{\frac{2\mu xy}{(x^2+y^2)^2}\zeta\right\} = 0$$

which reduces to

$$(x^2-y^2)\frac{\partial \zeta}{\partial x} + 2xy\frac{\partial \zeta}{\partial y} = 0 \tag{2.1}$$

of which the complete solution is

$$\zeta = \chi\{y/(x^2+y^2)\}, \tag{2.2}$$

χ being an arbitrary function.

The condition of integrability of the equations of rotational motion assumed steady is

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \nu\nabla^2\right)\zeta = 0,$$

or substituting for ζ from (2.2)

$$v\chi'\{y/(x^2+y^2)\}\,(x^2-y^2) - u\chi'\{y/(x^2+y^2)\}\;2xy = \nu\chi''\{y/(x^2+y^2)\},$$

and substituting for u and v in terms of the stream function ψ

$$(x^2-y^2)\frac{\partial \psi}{\partial x} + 2xy\frac{\partial \psi}{\partial y} = \frac{\nu\chi''\{y/(x^2+y^2)\}}{\chi'\{y/(x^2+y^2)\}}.$$

To solve this, the auxiliary equations are

$$\frac{dx}{x^2-y^2} = \frac{dy}{2xy} = \frac{\chi'\{y/(x^2+y^2)\}\,d\psi}{\nu\chi''\{y/(x^2+y^2)\}}$$

of which two solutions are

$$\frac{y}{(x^2+y^2)} = \text{constant, and}$$

$$\frac{\chi'\{y/(x^2+y^2)\}}{\chi''\{y/(x^2+y^2)\}}\psi + \frac{\nu x}{x^2+y^2} = \text{constant.}$$

Hence the complete solution is

$$\psi + \frac{\nu\chi''\{y/(x^2+y^2)\}}{\chi'\{y/(x^2+y^2)\}}\cdot\frac{x}{x^2+y^2} = G\{y/(x^2+y^2)\}, \tag{2.3}$$

where G is another arbitrary function.

As before this value of ψ has to fulfil the requirement

$$\nabla^2\psi = \zeta = \chi\{y/(x^2+y^2)\}. \tag{2.4}$$

Transforming to polar coordinates the value of ψ as obtained in (2.3) we get

$$\psi + \nu F\left(\frac{\sin\theta}{r}\right)\frac{\cos\theta}{r} = G\left(\frac{\sin\theta}{r}\right), \tag{2.5}$$

where

$$F = \frac{\chi''\left(\frac{\sin\theta}{r}\right)}{\chi'\left(\frac{\sin\theta}{r}\right)} .$$

Writing (2.4) as

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = \chi\left(\frac{\sin\theta}{r}\right)$$

and substituting ψ from (2.5), we get

$$\frac{G''}{r^4} - \frac{\nu\cos\theta\, F''}{r^5} = \chi\left(\frac{\sin\theta}{r}\right).$$

Since the only term containing $\cos\theta$ is

$$-\frac{\nu\cos\theta\, F''}{r^5}, \text{ we have}$$

$$F''\left(\frac{\sin\theta}{r}\right) = 0, \text{ and thereafter } G'' = 0,$$

so that $\chi = 0$, or the vorticity of the flow superposable on the doublet is zero, that is, the flow is irrotational.

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ON THE STRESSES DUE TO A SMALL SPHERICAL INCLUSION IN A UNIFORM BEAM UNDER CONSTANT BENDING MOMENT

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The concentration of stresses in the neighbourhood of a spherical inclusion under uniform tension has been obtained by Goodier (1933). The distribution of stresses round spherical and spheroidal inclusions in the case of an uniform state of stress infinitely far from the inclusion has been obtained by Edwards (1951). Stresses have also been calculated by Das (1952) in the case of similar inclusions on the axis of a large shaft which has been twisted by torsional stresses.

The object of this paper is to find the concentration of stresses due to a small spherical inclusion situated on the neutral axis of uniform beam under constant bending moment. The particular cases of a perfectly rigid inclusion and when the Young's modulus of the inclusion is one-quarter that of the surrounding material have been considered in detail.

1. Method of Solution.

The line of controids of the normal sections of the beam in the unstressed state has been taken as the z-axis. The x-axis is taken at right angles to this line in the plane in which the line of centroids is bent and the y-axis is drawn perpendicular to the axes of z and x.

When the beam is subjected to a constant bending moment only we can assume all the components of stresses to be zero except Z_z which is given by

$$Z_z = Ax \tag{1.1}$$

where A is a constant depending on the dimensions of the section and the curvature of the bent neutral axis. Taking the centre of the inclusion which is situated on the neutral axis as the origin and using polar co-ordinates such that

$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

and

$$z = r \cos\theta$$

we find that

$$Z_z = Ar \sin\theta \cos\varphi. \tag{1.2}$$

The stress components in polar co-ordinates corresponding to the above state can be put as

$$\widehat{rr} = \frac{Ar}{15}\,[3P_1^1(\cos\theta)+2P_3^1(\cos\theta)],$$

$$\widehat{\theta\theta} = \frac{2Ar}{15}\,[6P_1^1(\cos\theta)-P_3^1(\cos\theta)]\cos\varphi, \tag{1.3}$$

$$\widehat{r\theta} = \frac{2Ar}{45}\left[\frac{dP_3^1\,(\cos\theta)}{d\theta}-\frac{6dP_1^1\,(\cos\theta)}{d\theta}\right]\cos\varphi$$

and

$$\widehat{r\theta} = \widehat{\theta\varphi} = \widehat{\varphi\varphi} = 0$$

where $P_m^n(\cos\theta)$ is a Legendre's associated function of the first kind of the n-th order and the m-th degree.

The equations of equilibrium in polar co-ordinates are

$$(\lambda+2\mu)\sin\theta\frac{\partial\Delta}{\partial\theta}-2\mu\left[\frac{\partial\varpi_r}{\partial\varphi}-\frac{\partial}{\partial r}(r\varpi_\varphi\sin\theta)\right]=0$$

$$(\lambda+2\mu)\,\frac{1}{\sin\theta}\frac{\partial\Delta}{\partial\varphi}-2\mu\left[\frac{\partial}{\partial r}(r\varpi_\theta)-\frac{\partial\varpi_r}{\partial\theta}\right]=0$$

and

$$(\lambda+2\mu)r\sin\theta\frac{\partial\Delta}{\partial r}-2\mu\left[\frac{\partial}{\partial\theta}(\varpi_\varphi\sin\theta)-\frac{\partial\varpi_\theta}{\partial\varphi}\right]=0 \tag{1.4}$$

where

$$\Delta = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2u\sin\theta)+\frac{\partial}{\partial\theta}(rv\sin\theta)+\frac{\partial}{\partial\varphi}(rw)\right],$$

$$2\varpi_r = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(rw\sin\theta)-\frac{\partial}{\partial\varphi}(rv)\right],$$

$$2\varpi_\theta = \frac{1}{r\sin\theta}\left[\frac{\partial u}{\partial\varphi}-\frac{\partial}{\partial r}(rw\sin\theta)\right] \tag{1.5}$$

and

$$2\varpi_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rv)-\frac{\partial u}{\partial\theta}\right],$$

u, v, w being the components of displacement in the directions in which r, θ and φ increase respectively.

As particular solutions of the equations (1.4) we take

$$\Delta = [A_1rP_1^1(\cos\theta)+\frac{A_3}{r^4}\,P_3^1(\cos\theta)]\cos\varphi,$$

$$2\varpi_r = \left[B_2r+\frac{c_2}{r^4}\right]P_2^1(\cos\theta)\sin\varphi,$$

$$2\varpi_\theta = \left[-\frac{A_1 \alpha r}{2} + \frac{A_3 \alpha}{3r^4} \frac{P_3^1 (\cos\theta)}{\sin\theta} + \left(\frac{B_2 r}{2} - \frac{C_2}{3r^4} \right) \frac{dP_2^1 (\cos\theta)}{d\theta} \right] \sin\varphi,$$

and

$$2\varpi_\varphi = \left[-\frac{A_1 \alpha r}{2} \frac{dP_1^1 (\cos\theta)}{d\theta} + \frac{A_3 \alpha}{3r^4} \frac{dP_3^1 (\cos\theta)}{d\theta} \right.$$

$$\left. + \left(\frac{B_2 r}{2} - \frac{C_2}{3r^4} \right) \frac{P_2^1 (\cos\theta)}{\sin\theta} \right] \cos\varphi \qquad (1.6)$$

where $$\alpha = \lambda + 2\mu/\mu)$$

Assuming the inclusion to be very small compared with the dimensions of the beam, our problem is to find the components of displacement u_1, v_1, w_1 that will satisfy the equations of equilibrium given by (1.4) end will yield stresses given by (1,3) when $r \to \infty$. (1.7)

Also for the inclusion the components of displacement u_2, v_2, w_2 must be such that

$$u_1 = u_2, \qquad (1.8)$$

$$v_1 = v_2$$

and $$w_1 = w_2$$

when $r = a$, a being the radius of the spherical inclusion.

Further when $r = a$. we must have

$$[\widehat{rr}]_1 = [\widehat{rr}]_2$$

$$[\widehat{r\theta}]_1 = [\widehat{r\theta}]_2 \qquad (1.9)$$

$$[\widehat{r\varphi}]_1 = [\widehat{r\varphi}]_2$$

where suffixes 1 and 2 denote stresses calculated from u_1, v_1, w_1 and u_2, v_2, w_2 respectively.

2. Solution of the Problem.

The components of displacement satisfying (1.6) can be taken as when $r \geq a$

$$u_1 = \left[\frac{A_1(3-\alpha)}{10} r^2 - \frac{2E_1}{r^3} \right] P_1^1 (\cos\theta) \cos\varphi$$

$$+ \left[\frac{A_3(1+2\alpha)}{5r^3} + 3D_3 r^3 - \frac{4E_3}{r^5} \right] P (\cos\theta) \cos\varphi$$

$$v_1 = \left[\frac{A_1(1-2\alpha)}{10} r^3 + \frac{E_1}{r^3} \right] \frac{dP_1^1 (\cos\theta)}{d\theta} \cos\varphi + \left[\frac{B_2 r^2}{6} + \frac{C_2}{6r^3} \right] \frac{P_2^1 (\cos\theta)}{\sin\theta} \cdot \cos\varphi$$

$$+ \left[D_3 r^2 + \frac{E^3}{r^5} - \frac{A_3(3-\alpha)}{30r^3} \right] \frac{dP_3^1 (\cos\theta)}{d\theta} \cos\varphi$$

$$w_1=\left[-\frac{A_1(1-2\alpha)}{10}r^2-\frac{E_1}{r^3}\right]\frac{P_1^1(\cos\theta)}{\sin\theta}\sin\varphi-\left[\frac{B_2r^2}{6}+\frac{C_2}{6r^3}\right]\frac{dP_2^1(\cos\theta)}{d\theta}\sin\varphi \tag{2.1}$$

$$+\left[-D_3r^2-\frac{E_3}{r^5}+\frac{A_3(3-\alpha)}{30r^3}\right]\frac{P_3^1(\cos\theta)}{\sin\theta}\sin\varphi$$

and when $r \leq a$ *i.e.* within the inclusion,

$$u_2=\left[\frac{A_1'(3-\alpha')}{10}r^2\right]P_1^1(\cos\theta)\cos\varphi+3D_3'r^2P_3^1(\cos\theta)\cos\varphi$$

$$v_2=\left[\frac{A_1'(3-\alpha')}{10}r^2\right]\frac{dP_1^1(\cos\theta)}{d\theta}\cos\varphi+\frac{B_2'r^2}{6}\frac{P_2^1(\cos\theta)}{\sin\theta}\cos\varphi+D_3'r^2\frac{dP_3^1(\cos\theta)}{d\theta}\cos\varphi$$

$$w_2=\left[-\frac{A_1'(1-2\alpha')}{10}r^2\right]\frac{P_1^1(\cos\theta)}{\sin\theta}\sin\varphi-\frac{B_2'r^2}{6}\frac{dP_2^1(\cos\theta)}{d\theta}\sin\varphi-D_3'r^2\frac{P_3^1(\cos\theta)}{\sin\theta}\sin\varphi \tag{2.2}$$

where $\alpha'=(\lambda'+2\mu')/\mu'$; λ', μ' being elastic constants for the inclusion.

The stresses calculated from the displacements (2.1) are given below:

$$\frac{\widehat{rr}}{\cos\varphi}=\left[\frac{A_1(3\lambda+2\mu)}{5}r+\frac{12\mu E_1}{r^4}\right]P_1^1(\cos\theta)+\left[\frac{A_3(17\lambda+18\mu)}{5r^4}+12\mu D_3r+\frac{40\mu E_3}{r6}\right]P_3^1(\cos\theta)$$

$$\frac{\widehat{r\theta}}{\cos\varphi}=\left[-\frac{A_1(3\lambda+2\mu)}{10}r-\frac{6\mu E_1}{r^4}\right]\frac{dP_1^1(\cos\theta)}{d\theta}$$

$$+\left[\frac{\mu B_2r}{6}-\frac{2\mu C_2}{3r^4}\right]\frac{P_2^1(\cos\theta)}{\sin\theta}+\left[4\mu D_3r-\frac{10\mu E_3}{r^6}-\frac{A_3(8\lambda+7\mu)}{15r^4}\right]\frac{dP_3^1(\cos\theta)}{d\theta}$$

$$\frac{\widehat{r\varphi}}{\sin\varphi}=\left[\frac{A_1(3\lambda+2\mu)}{10}+\frac{6\mu E_1}{r^4}\right]\frac{P_1^1(\cos\theta)}{\sin\theta}$$

$$-\left[\frac{\mu B_2r}{6}-\frac{2\mu C_2}{3r^4}\right]\frac{dP_2^1(\cos\theta)}{d\theta}+\left[-4\mu D_3r+\frac{10\mu E_3}{r^6}+\frac{A_3(8\lambda+7\mu)}{15r^4}\right]\frac{P_3^1(\cos\theta)}{\sin\theta}$$

$$\frac{\widehat{\theta\theta}}{\cos\varphi}=\left[\left\{\frac{2A_1(3\lambda+2\mu)}{5}-\mu B_2+6\mu D_3\right\}r\right.$$

$$\left.+\left\{\frac{A_3(\lambda-\mu)}{5}-6\mu E_1-\mu C_2\right\}\frac{1}{r^4}+\frac{6\mu E_3}{r^6}\right]P_1^1(\cos\theta)-\left[12\mu D_3r+\frac{26\mu E_3}{r^6}+\frac{A_3(2\lambda+3\mu)}{5r^4}\right]P_3^1(\cos\theta)$$

$$\frac{\widehat{\varphi\varphi}}{\cos\varphi}=\left[\left\{\frac{2A_1(3\lambda+2\mu)}{5}+\mu B_2-6\mu D_3\right\}r\right.$$

$$\left.-\left\{\frac{A_3(\lambda-\mu)}{5}+6\mu E_1-\mu C_2\right\}\frac{1}{r^4}-\frac{6\mu E_3}{r^6}\right]P_1^1(\cos\theta)-\left[\frac{A_3\mu}{r^4}+\frac{14\mu E_3}{r^6}\right]P_3^1(\cos\theta)$$

$$\frac{\widehat{\theta\varphi}}{\sin\varphi}=\mu\left[(B_2+30D_3)r+\{C_2-A_3(3-\alpha)\}\frac{1}{r^4}+\frac{30E_3}{r^6}\right]\sin\theta\cos\theta \tag{2·3}$$

The stresses calculated from (2.2) are as follows:

$$\frac{\widehat{rr}}{\cos\varphi}=\frac{A_1'(3\lambda'+2\mu')}{5}\,rP_1^1(\cos\theta)+12\mu'D_3'r\,P_3^1(\cos\theta)$$

$$\frac{\widehat{r\theta}}{\cos\varphi}=-\frac{A_1'(3\lambda'+2\mu')}{10}\,\frac{dP_1^1(\cos\theta)}{d\theta}+\frac{\mu'B_2'r}{6}\,\frac{P_2^1(\cos\theta)}{\sin\theta}+4\mu'D_3'r\frac{dP_3^1(\cos\theta)}{d\theta}$$

$$\frac{\widehat{r\varphi}}{\sin\varphi}=\frac{A_1'(3\lambda'+2\mu')}{10}\,r\,\frac{P_1^1(\cos\theta)}{\sin\theta}-\frac{\mu'B_2'}{6}\,r\,\frac{dP_2^1(\cos\theta)}{d\theta}-4\mu'D_3'r\,\frac{P_3^1(\cos\theta)}{\sin\theta}$$

$$\frac{\widehat{\theta\theta}}{\cos\varphi}=\left[\frac{2A_1'(3\lambda'+2\mu')}{5}-\mu B_2'+6\mu'D_3'\right]rP_1^1(\cos\theta)-12\mu'D_3'\,rP_3^1(\cos\theta)$$

$$\frac{\widehat{\varphi\varphi}}{\cos\varphi}=\left[\frac{2A_1'(3\lambda'+2\mu')}{5}+\mu'B_2'-6\mu'D_3'\right]rP_1^1(\cos\theta)$$

$$\frac{\widehat{\theta\varphi}}{\sin\varphi}=\mu'\,[B_2'+30D_3']r\,\sin\theta\cos\theta \tag{2.4)}$$

The conditions (1.7), (1.1) and (1.9) will be satisfied if

$$A_1=\frac{A}{3\lambda+2\mu},$$

$$B_2=-\frac{A}{3\mu},$$

$$D_3=\frac{A}{90\mu},$$

$$A_3=\frac{14A(\mu'-\mu)a^5}{3[\mu(19\lambda+26\mu)+4\mu'(5\lambda+11\mu)]},$$

$$E_3=\frac{-A(\lambda+\mu)(\mu'-\mu)a^7}{3\mu[\mu(19\lambda+26\mu)+4\mu'(4\lambda+11\mu)]},$$

$$D_3'=\frac{7A(\lambda+2\mu)}{18[\mu(19\lambda+26\mu)+4\mu'(4\lambda+11\mu)]},$$

$$E_1=\frac{A_1a^5}{20\mu}\left[\frac{\mu'(\mu-\lambda)(3\lambda'+2\mu')-\mu(\mu'-\lambda')(3\lambda+2\mu)}{3\mu(\mu'-\lambda')+\mu'(3\lambda'+2\mu')}\right]$$

$$A_2'=A_1\,\frac{5\mu\mu'}{3\mu(\mu'-\lambda')+\mu'(3\lambda'+2\mu')}$$

$$B_2'=\frac{2A_1}{5(4\mu+\mu')}\left[\frac{\mu'(\mu-\lambda)(3\lambda'+2\mu')-\mu(\mu'-\lambda')(3\lambda+2\mu)+10\mu^2(2\lambda'+3\mu')}{3\mu(\mu'-\lambda')+\mu'(3\lambda'+2\mu')}-\frac{99\lambda+86\mu}{15}\right],$$

$$C_2=-\frac{A_1a^5\mu'}{10\mu(4\mu+\mu')}\left[\frac{\mu'(\mu-\lambda)(3\lambda'+2\mu')-\mu(\mu'-\lambda')(3\lambda+2\mu)+10\mu^2(2\lambda'+3\mu')}{3\mu(\mu'-\lambda')+\mu'(3\lambda'+2\mu')}\right.$$
$$\left.-\frac{2\{\mu'(21\lambda+19\mu)-5\mu(3\lambda+2\mu)\}}{3\mu'}\right] \quad (2.5)$$

Hence on the surface the stresses become

$$\left(\frac{\widehat{rr}}{\cos\varphi}\right)_{r=a}=\left[\frac{(3\lambda'+2\mu')\,\mu}{(3\lambda+2\mu)Q}+\frac{84(\lambda+2\mu)}{P}\right]A\mu' a\sin\theta-\left[\frac{105(\lambda+2\mu)}{P}\right]A\mu' a\sin^3\theta$$

$$\left(\frac{\widehat{r\theta}}{\cos\varphi}\right)_{r=a}=\left[\frac{5\mu\{4\mu(\mu'-\lambda')-\mu'(3\lambda'+2\mu')\}+2R}{10(4\mu+\mu')(3\lambda+2\mu)Q}-\frac{(99\lambda+86\mu)}{75(4\mu+\mu')(3\lambda+2\mu)}\right.$$
$$\left.-\frac{77(\lambda+2\mu)}{P}\right]\mu' Aa\cos\theta+\frac{105(\lambda+2\mu)\mu'}{P}Aa\cos^3\theta$$

$$\left(\frac{\widehat{r\varphi}}{\sin\varphi}\right)_{r=a}=\left[\frac{5\mu\{\mu'(3\lambda'+2\mu'-4\mu(\mu'-\lambda')\}-2R}{10(4\mu+\mu')(3\lambda+2\mu)\;Q}+\frac{(99\lambda+86\mu)}{75(4\mu+\mu')(3\lambda+2\mu)}-\frac{28(\lambda+2\mu)}{P}\mu' Aa\right.$$
$$+\left[\frac{2R+20\mu^2(2\lambda'+3\mu')}{5(4\mu+\mu')(3\lambda+2\mu)Q}+\frac{35(\lambda+2\mu)}{P}-\frac{2(99\mu+86\mu)}{75(4\mu+\mu')(3\lambda+3\mu)\,Q}\right]\mu' Aa\sin^3\theta$$

$$\left(\frac{\widehat{\theta\theta}}{\cos\varphi}\right)_{r=a}=\frac{(\mu'-\mu)(596\lambda+484\mu)}{5P}-\frac{(\mu'+6\mu)\;R-5\mu^2\mu'(2\lambda'+3\mu')}{5Q(\mu'+4\mu)(3\lambda+2\mu)}$$
$$\left.+\frac{5\mu(3\lambda+2\mu)-\mu'(21\lambda+19\mu)}{15(\mu'+4\mu)(3\lambda+2\mu)}\right]Aa\sin\theta-\left[\frac{3(51\lambda+44\mu)(\mu'-\mu)}{P}-1\right]Aa\sin^3\theta$$

$$\left(\frac{\widehat{\varphi\varphi}}{\cos\varphi}\right)_{r=a}=\left[\frac{(\mu'-\mu)(436\lambda+484\mu)}{5P}-\frac{2(\mu'+3\mu)R+5\mu^2\mu'(2\lambda\;+3\mu')}{5Q(\mu'+4\mu)(3\lambda+2\mu)}\right.$$
$$\left.-\frac{5\mu(3\lambda+2\mu)-\mu'(21\lambda+19\mu)}{15(\mu'+4\mu)(3\lambda+2\mu)}\right]Aa\sin\theta-\left[\frac{105\lambda(\mu'+\mu)}{P}\right]Aa\sin^3\theta$$

$$\left(\frac{\widehat{\theta\varphi}}{\sin\varphi}\right)_{r=a}=\left[-\frac{(\mu'-\mu)(16\lambda+44\mu)}{P}-\frac{\mu'}{10(3\lambda+2\mu)(\mu'+4\mu)}\left\{\frac{\mathrm{R}+10\mu^2\,(2\lambda'+3\mu')}{Q}\right.\right.$$
$$\left.\left.-\frac{2(21\lambda+19\mu)}{3}+\frac{10\mu(3\lambda+2\mu)}{3\mu'}\right\}\right]Aa\sin\theta\cos\theta \quad (2{\cdot}5)$$

where $P=3[\mu(19\lambda+26\mu)+4\mu'(4\lambda+11\mu)]$

$Q=3\mu(\mu'-\lambda')+\mu'(3\lambda'+2\mu')$

and $R=\mu'(\mu-\lambda)(3\lambda'+2\mu')-\mu(\mu'-\lambda')(3\lambda+2\mu)$

To obtain the stresses when there is a flaw instead af an inclusion we put $\mu=0$ in the above stresses and get

$$\left(\frac{\widehat{\theta\theta}}{\cos\varphi}\right)_{r=a}=-\frac{5(29\lambda+22\mu)}{4(19\lambda+26\mu)}Aa\sin\theta+\frac{70(\lambda+\mu)}{19\lambda+26\mu}Aa\sin^3\theta$$

$$\left(\frac{\widehat{\varphi\varphi}}{\cos\varphi}\right)_{r=a}=-\frac{5(23\lambda+2\mu)}{4(19\lambda+26\mu)}Aa\sin\theta+\frac{35\lambda}{19\lambda+26\mu}Aa\sin^3\theta \tag{2·6}$$

$$\left(\frac{\widehat{\theta\varphi}}{\sin\varphi}\right)_{r=a}=\frac{5(3\lambda+10\mu)}{4(19\lambda+26\mu)}Aa\sin\theta\cos\theta,$$

$$(\widehat{r\,r})_{r=a}=(\widehat{r\,\varphi})_{r=a}=(\widehat{r\,\theta})_{r=a}=0$$

The above results were obtained by Sen (1933).

We now consider the case when the inclusion is perfectly rigid. Here we make $\mu'\rightarrow\infty$ and get

$$\left(\frac{\widehat{r\,r}}{\cos\varphi}\right)_{r=a}=\left[\frac{\mu}{(3\lambda+2\mu)}+\frac{7(\lambda+2\mu)}{(4\lambda+11\mu)}\right]Aa\sin\theta-\frac{35}{4}\frac{(\lambda+2\mu)}{(4\lambda+11\lambda)}Aa\sin^3\theta$$

$$\left(\frac{\widehat{r\,\theta}}{\cos\varphi}\right)_{r=a}=\left[-\frac{(228\lambda+217\mu)}{150(3\lambda+2\mu)}-\frac{77(\lambda+2\mu)}{12(4\lambda+11\mu)}\right]Aa\cos\theta+\frac{35}{4}\frac{(\lambda+2\mu)}{(4\lambda+11\mu)}Aa\cos^3\theta,$$

$$\left(\frac{\widehat{r\,\varphi}}{\sin\varphi}\right)_{r=a}=\left[\frac{(228\lambda+217\mu)}{150(3\lambda+2\mu)}-\frac{7(\lambda+2\mu)}{3(4\lambda+11\mu)}\right]Aa+\left[\frac{35}{12}\frac{(\lambda+2\mu)}{(4\lambda+11\mu)}-\frac{2(114\lambda+71\mu)}{75(3\lambda+2\mu)}\right]Aa\sin^2\theta$$

$$\left(\frac{\widehat{\theta\,\theta}}{\cos\varphi}\right)_{r=a}=\left[\frac{149\lambda+121\mu}{15(4\lambda+11\mu)}-\frac{18\lambda+22\mu}{15(3\lambda+2\mu)}\right]Aa\sin\theta-\left[\frac{51\lambda+44\mu}{4(4\lambda+11\mu)}-1\right]Aa\sin^3\theta$$

$$\left(\frac{\widehat{\varphi\varphi}}{\cos\varphi}\right)_{r=a}=\left[\frac{109\lambda+11\mu}{15(4\lambda+11\mu)}+\frac{27\lambda+18\mu}{15(3\lambda+2\mu)}\right]Aa\sin\theta-\frac{35\lambda}{4(4\lambda+11\mu)}Aa\sin^3\theta$$

$$\left(\frac{\widehat{\theta\varphi}}{\sin\varphi}\right)_{r=a}=\frac{\lambda+\mu}{2(3\lambda+2\mu)}Aa\sin\theta\cos\theta$$

If we take $\lambda=\mu$ or Poisson's ratio σ of the material $=\frac{1}{4}$, the above sresses become

$$\left(\frac{\widehat{rr}}{\cos\varphi}\right)_{r=a}=\tfrac{8}{5}Aa\sin\theta-\tfrac{7}{4}Aa\sin^3\theta$$

$$\left(\frac{\widehat{r\theta}}{\cos\varphi}\right)_{r=a}=-\tfrac{563}{300}Aa\cos\theta+\tfrac{7}{4}Aa\cos^3\theta$$

$$\left(\frac{\widehat{r\,\varphi}}{\sin\varphi}\right)_{r=a}=\tfrac{19}{150}Aa-\tfrac{121}{300}Aa\sin^2\theta$$

$$\left(\frac{\widehat{\theta\theta}}{\cos\varphi}\right)_{r=a}=\tfrac{2}{3}Aa\sin\theta-\tfrac{7}{12}Aa\sin^3\theta$$

$$\left(\frac{\widehat{\varphi\varphi}}{\cos\varphi}\right)_{r=a}=\tfrac{16}{15}Aa\sin\theta-\tfrac{7}{12}Aa\sin^3\theta \tag{2.8}$$

$$\left(\frac{\widehat{\theta\varphi}}{\sin\varphi}\right)_{r=a}=\tfrac{1}{5}Aa\sin\theta\cos\theta$$

Now putting $A = -M/I$ where M is the bending moment about y-axis and I the moment of inertia of a section about an axis through its centre of gravity parallel to the y-axis we get the numerically greatest value of the above stresses given by

$$\left(\frac{\widehat{rr}}{\cos\varphi}\right)_{r=a} = 0{\cdot}59\,\frac{M}{I}a, \quad \text{when } \theta = -35{\cdot}5^\circ$$

$$\left(\frac{\widehat{r\theta}}{\cos\varphi}\right)_{r=a} = 0{\cdot}75\,\frac{M}{I}a, \quad \text{when } \theta = 53^\circ$$

$$\left(\frac{\widehat{r\varphi}}{\sin\varphi}\right)_{r=a} = 0{\cdot}28\,\frac{M}{I}a, \quad \text{when } \theta = 90^\circ$$

$$\left(\frac{\widehat{\theta\theta}}{\cos\varphi}\right)_{r=a} = 0{\cdot}27\,\frac{M}{I}a, \quad \text{when } \theta = -38^\circ$$

$$\left(\frac{\widehat{\varphi\varphi}}{\cos\varphi}\right)_{r=a} = 0{\cdot}55\,\frac{M}{I}a, \quad \text{when } \theta = -51{\cdot}3^\circ \tag{2.9}$$

$$\left(\frac{\widehat{\theta\varphi}}{\sin\varphi}\right)_{r=a} = 0{\cdot}1\,\frac{M}{I}a, \quad \text{when } \theta = -45^\circ$$

The above stresses when there is no inclusion are given by

$$\left(\frac{\widehat{rr}}{\cos\varphi}\right)_{r=a} = 0{\cdot}46\,\frac{M}{I}a, \quad \text{when } \theta = -33{\cdot}5^\circ$$

$$\left(\frac{\widehat{r\theta}}{\cos\varphi}\right)_{r=a} = 0{\cdot}25\,\frac{M}{I}a, \quad \text{when } \theta = 53^\circ$$

$$\left(\frac{\widehat{\theta\theta}}{\cos\varphi}\right)_{r=a} = 0{\cdot}23\,\frac{M}{I}a, \quad \text{when } \theta = -38^\circ \tag{2.10}$$

and

$$(\widehat{r\varphi})_{r=a} = (\widehat{\varphi\varphi})_{r=a} = (\widehat{\theta\varphi})_{r=a} = 0$$

If the inclusion consists of a material having the same Poisson's ratio ¼ but an Young's Modulus one-quarter of that of the surrounding metal taken as a plausible value for slag in steel the stresses are given by

$$\left(\frac{\widehat{rr}}{\cos\varphi}\right)_{r=a} = \frac{11}{20}Aa\sin\theta - \frac{7}{16}Aa\sin^3\theta$$

$$\left(\frac{\widehat{r\theta}}{\cos\varphi}\right)_{r=a} = -\frac{9}{51}Aa\cos\theta + \frac{7}{16}Aa\cos^3\theta$$

$$\left(\frac{\widehat{r\varphi}}{\sin\varphi}\right)_{r=a} = -\frac{4}{51}Aa + \frac{14}{51}Aa\sin^2\theta \tag{2·11}$$

$$\left(\frac{\widehat{\theta\theta}}{\cos\varphi}\right)_{r=a} = -\frac{14}{17}Aa\sin\theta + \frac{35}{16}Aa\sin^3\theta$$

$$\left(\frac{\widehat{\varphi\varphi}}{\cos\varphi}\right)_{r=a} = -\frac{1}{2}Aa\sin\theta + \frac{7}{16}Aa\sin^3\theta$$

$$\left(\frac{\widehat{\theta\varphi}}{\sin\varphi}\right)_{r=a} = \frac{54}{340}Aa\sin\theta\cos\theta$$

Proceeding as in the previous case the numerically greatest values for the above stresses can be calculated to be

$$\left(\frac{\widehat{rr}}{\cos\varphi}\right)_{r=a} = 0{\cdot}24\frac{M}{I}a, \quad \text{when} \quad \theta = 40{\cdot}3^\circ \tag{2.12}$$

$$\left(\frac{\widehat{r\theta}}{\cos\varphi}\right)_{r=a} = 0{\cdot}26\frac{M}{I}a, \quad \text{when} \quad \theta = 180^\circ$$

$$\left(\frac{\widehat{r\phi}}{\sin\varphi}\right)_{r=a} = -0{\cdot}20\frac{M}{I}a, \quad \text{when} \quad \theta = 90^\circ$$

$$\left(\frac{\widehat{\theta\theta}}{\cos\varphi}\right)_{r=a} = 1{\cdot}86\frac{M}{I}a, \quad \text{when} \quad \theta = -90^\circ$$

$$\left(\frac{\widehat{\varphi\varphi}}{\cos\varphi}\right)_{r=a} = 0{\cdot}206\frac{M}{I}a, \quad \text{when} \quad \theta = 38^\circ \qquad 2.12$$

$$\left(\frac{\widehat{\theta\varphi}}{\sin\varphi}\right)_{r=a} = 0{\cdot}078\frac{M}{I}a, \quad \text{when} \quad \theta = -45^\circ$$

Also here

$$\left(\frac{\widehat{\theta\theta}-\widehat{\varphi\varphi}}{\cos\varphi}\right)_{r=a} = -\frac{11}{84}Aa\sin\theta + \frac{7}{4}Aa\sin^3\theta \tag{2·13}$$

This will have the numerically greatest value when $\theta = -\pi/2$ and the value is given by

$$\frac{97}{68}\frac{M}{I}a \tag{2.14}$$

The numerically greatest value of the difference of the tensions are obtained at the point

$$\varphi = 0,\ \theta = -\pi/2 \quad \text{or} \quad \varphi = \pi,\ \theta = \pi/2$$

and if fracture occurs, it commences at that point.

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THE EFFECT OF TWO EQUAL CIRCULAR HOLES ON THE STRESS DISTRIBUTION IN A BEAM UNDER UNIFORM BENDING MOMENT

By

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1. Introduction

The problem of a circular hole with its centre at the neutral axis of a beam under uniform bending moment has been discussed and the result obtained have been verified approximately by photo-elastic measurement by Z. Tuzi (1928). The object of this paper is to study the influence on the stress distribution of two equal circular holes with their centres at the neutral axis of a deep beam of rectangular cross section under uniform bending moment M. For this purpose bipolar coordinates are introduced.

2. Bipolar coordinates and the expression for the stresses.

Let us introduce the curvilinear coordinates defined by

$$\alpha + i\beta = \log_e \frac{x+i(y+a)}{x+i(y-a)}, \tag{1}$$

where x, y are cartesian coordinates and a is a positive length.

Solving for x and y we have,

$$x = \frac{a \sin \beta}{\cosh \alpha - \cos \beta}, \qquad y = \frac{a \sinh \alpha}{\cosh \alpha - \cos \beta}. \tag{2}$$

From the relation

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2$$

we get,

$$h = (\cosh \alpha - \cosh \beta)/a \tag{3}$$

The curves $\alpha =$ constant are a set of coaxial circles having the two poles $(0, \pm a)$ for the limiting points, and $\beta =$ constant another set of circles passing through the limiting points, $(0, \pm a)$ and intersecting the first set orthogonally. The axis of x is given by $\alpha = 0$. The radius of a circle α is a cosech α, while the coordinates of its centre are $(0, a \coth \alpha)$ We have $\beta = 0$ on the y axis, except on the segment joining the limiting points $(0, \pm a)$ where $\beta = \pm\pi$.

At infinity $\alpha = 0$, $\beta = 0$ and at the limiting points $\alpha = \pm\infty$.

The stresses in terms of the stress function χ are given by (Jeffery 1921, p. 269) .

$$a\widehat{\alpha\alpha} = \left\{(\cosh \alpha - \cos \beta)\frac{\partial^2}{\partial \beta^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cosh \alpha\right\}(h\chi),$$

$$a\widehat{\beta\beta} = \left\{(\cosh \alpha - \cos \beta)\frac{\partial^2}{\partial \alpha^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cos \beta\right\}(h\chi),$$

$$a\widehat{\alpha\beta} = -(\cosh \alpha - \cos \beta)\frac{\partial^2}{\partial \alpha \partial \beta}(h\chi). \tag{4}$$

So that, we have

$$a(\widehat{\alpha\alpha}-\widehat{\beta\beta}) = (\cosh\alpha-\cos\beta)\left\{\frac{\partial^2}{\partial\beta^2}-\frac{\partial^2}{\partial\alpha^2}+1\right\}(h\chi). \tag{5}$$

The necessary and sufficient conditions for a boundary α = constant, to be free from stress are (Jeffery 1921, p. 280) that on the boundary

$$\frac{\partial}{\partial\alpha}(h\chi) = \text{constant} = \varrho \text{ (say)} \tag{6}$$

and

$$h\chi = \varrho\tanh\alpha+\sigma(\cosh\alpha\cos\beta-1)+\tau\sin\beta, \tag{7}$$

where ϱ, σ, τ are Michell's three constants of the boundary.

3. The effect of two equal circular holes on the stress distribution in a beam under uniform bending moment M.

Let the holes be defined by $\alpha = c\ (>0)$ and $\alpha = -c$. The origin of coordinates and the centres of the holes being at the neutral axis.

For a beam of rectangular cross-section of thickness $2b$ and depth $2d$ subjected to a constant bending moment M, we have, the stress function

$$\chi_0 = \frac{A}{6}x^3 \tag{8}$$

giving $\widehat{xx} = \widehat{xy} = 0$, $\widehat{yy} = Ax$, where

$$A = \frac{3M}{4d^3 b}.$$

So that,

$$h\chi_0 = \frac{A}{6}\,\frac{a^3\sin^3\beta}{(\cosh\alpha-\cos\beta)^3} \tag{9}$$

and this has different expansions on different sides of the line $\alpha = 0$. Thus for $\alpha > 0$

$$h\chi_0 = \tfrac{1}{2}A\,a^2e^{-2\alpha}\sin\beta+\tfrac{1}{3}A\,a^2\sum_{n=2}^{\infty}(2\cosh\alpha-n\sinh\alpha)\,e^{-n\alpha}\sin n\beta, \tag{10}$$

and for $\alpha < 0$

$$h\chi_0 = \tfrac{1}{2}A\,a^2e^{2\alpha}\sin\beta+\tfrac{1}{3}A\,a^2\sum_{n=2}^{\infty}(2\cosh\alpha+n\sinh\alpha)\,e^{n\alpha}\sin n\beta. \tag{11}$$

For the complete stress function we have to add to χ_0 a stress function χ_1 which gives no stress at infinity and is such that the complete stress function $(\chi_0+\chi_1)$ will give no stress over $\alpha = \pm c$.

We assume

$$h\chi_1 = C_1\cosh 2\alpha\sin\beta+\sum_{n=2}^{\infty}\left[C_n\cosh(n+1)\alpha+D_n\cosh(n-1)\alpha\right]\sin n\beta. \tag{12}$$

so that the stresses due to χ_1 at infinity vanish.

The complete stress function $\chi = \chi_0+\chi_1$ given by (10) and (12) satisfies the boundary conditions (6) and (7) for the boundary $\alpha = c$. So that if ϱ, σ, τ be the Michell's three constants of the boundary $\alpha = c$, we find without difficulty that $\varrho = \sigma = 0$,

$$C_1 \cosh 2c + \tfrac{1}{2}A\, a^2\, e^{-2c} = \tau,$$

$$2C_1 \sinh 2c - A\, a^2\, e^{-2c} = 0. \tag{13}$$

For $n \geq 2$

$$C_n \cosh (n+1)\, c + D_n \cosh (n-1)\, c = -\tfrac{1}{3}A\, a^2(2 \cosh c - n \sinh c)\, e^{-nc}, \tag{14}$$

$$(n+1)\ C_n \sinh (n+1)\, c + (n-1)D_n \sinh (n-1)c = -\tfrac{1}{3}A\, a^2[(n^2+2) \sinh c - 3n \cosh c]e^{-nc}.$$

For $\alpha = -c$, the stress function calculated from (11) and (12) satisfies the boundary conditions (6) and (7), so that we get the same system of equations (13) and (14) where ρ, σ, τ are now replaced by three other constants ρ', σ', τ'.

Solving equations (13) and (14), we get

$$C_1 = \tfrac{1}{2}A\, a^2\, e^{-2c} \operatorname{cosech} 2c \tag{15}$$

For $n \geq 2$

$$C_n [\sinh 2nc + n \sinh 2c] = \tfrac{1}{6}A\, a^2[n(n+2)\, e^{-2c} - (n-1)(n-2) + (n+2)e^{-2nc}], \tag{16}$$

and $\quad D_n[\sinh 2nc + n \sinh 2c] = \tfrac{1}{6}A\, a^2[n(n-2)\, e^{2c} - (n+1)(n+2) - (n-2)e^{-2nc}].$

On the hole $\alpha = c$, $\widehat{\alpha\alpha} = 0$ so that $\widehat{\beta\beta}$ can be very easily calculated from (5).

We have,

$$\widehat{\beta\beta}_c = 2A\, a(\cosh c - \cos \beta)[\operatorname{cosech} 2c \sin \beta - \tfrac{1}{3} \sum_{n=2}^{\infty} R_n \sin n\beta], \tag{17}$$

where

$$[R_n + n(n-2)e^{-(n-1)c}](\sinh 2nc + n \sinh 2c)$$
$$= n[n(n-2)e^{-2c} - (n+1)(n+2) - (n-2)e^{-2nc}]\cosh (n-1)c$$

The series in (17) is slowly convergent for small values of c. To make it rapidly convergent, we put

$$R_n = 2(n^3 \sinh c - 3n^2 \cosh c + 2n \sinh c)e^{-nc} + M_n \tag{18}$$

where,

$$M_n[\sinh 2nc + n \sinh 2c]$$
$$= ne^{-nc}\{[e^{2c} + e^{-2nc} - 2n \sinh 2c][(n^2+2) \sinh c - 3n \cosh c] - (n-1)(n-2)e^{c} \sinh 2c\}.$$

Remembering that

$$\tfrac{1}{3}(\cosh c - \cos \beta \sum_{n=1}^{\infty} 2n(n^2 \sinh c - 3n \cosh c + 2 \sinh c)e^{-nc} \sin n\beta$$

$$= \frac{\sin \beta}{(\cosh c - \cos \beta)^3}\, [\cosh 2c \cos^2\beta - 2 \cosh c \cos \beta + 1 - \sinh^2 c],$$

we have,

$$\widehat{\beta\beta}_c = 2Aa\Big\{\frac{\sin \beta\, (\cosh c - \cos \beta)}{e^{4c} \sinh 2c}$$

$$+ \frac{\sin \beta}{(\cosh c - \cos \beta)}\, [\sinh^2 c + 2 \cosh c \cos \beta - \cosh 2c \cos^2 \beta - 1]$$

$$- \tfrac{1}{3}\, (\cosh c - \cos \beta)\sum_{n=2}^{\infty} M_n \sin n\beta\Big\} \tag{19}$$

The table below gives the values of the stress $\widehat{\beta\beta}$ on the boundary of the hole $\alpha = c$ for positive values of β between 0 and π when $c = \cdot 8$. From the same table the values of $\widehat{\beta\beta}$ for negative values of β are obtained by changing the sign of $\widehat{\beta\beta}$.

Table.

$c = \cdot 8$.

β	$\widehat{\beta\beta}_c/Aa$	β	$\widehat{\beta\beta}_c/Aa$	β	$\widehat{\beta\beta}_c/Aa$
0°	0	60°	1·2952	120°	—0·243
15°	—0·3566	75°	0·4208	135°	—0·1636
30°	1·5742	90°	—0·0861	150°	—0·0886
45°	2·0938	105°	—0·2686	165°	—0·0298
				180°	0

For the purpose of comparison with the corresponding case of a single hole we introduce the angle θ between the radii of the points $(c, 0)$ and $(c, \beta,)$ on the boundary of the hole $\alpha = c$. So that

$$\cos\theta = \frac{\cosh c \cos\beta - 1}{\cosh c - \cos\beta} \cdot$$

It is found after calculation that the maximum stress $\pm$ 1·901 RA (R being the radius of the hole) takes place at $\beta = \pm 42°$ approximately *i.e.*, at $\theta = 90°36'$ approx. as compared with $\pm 2\ RA$ at $\pi/2$ in the case with a single hole There are six zero values of this stress at the points $\beta = 0$, π, $\pm 18°10'$ and $\pm 86°28'$ approx. *i e.*, at $\theta = 0$, π, $\pm 45°38'$ and $\pm 136°$ approx. respectively, instead of at $\theta = 0$, π, $\pm \frac{1}{4}\pi$ and $\pm \frac{3}{4}\pi$ in the case with a single hole.

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SOME INVERSION FORMULAE FOR THE GENERALISED HANKEL TRANSFORM

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1. **Introduction:** In a recent paper I gave a generalisation of the wellknown Hankel transform by solving the integral equation

$$f(x) = (\tfrac{1}{2})^{\lambda} \int_{0}^{\infty} (xy)^{\lambda+\frac{1}{2}} J_{\lambda}^{\mu}(\tfrac{1}{2}x^{2}y^{2}) g(y)\,dy \tag{1}$$

where

$$J_{\lambda}^{\mu}(x) = \sum_{r=0}^{\infty} \frac{(-x)^{r}}{r\,!\,\Gamma(1+\lambda+\mu r)}, \quad \mu > 0.$$

I showed that the reciprocal kernel was expressible in terms of a function of the type $J_{\lambda}^{\mu}(x)$. In fact it was proved that

$$g(y) = \mu^{-1} 2^{\lambda+2-2(1+\lambda)/\mu} \int_{0}^{\infty} (xy)^{2(1+\lambda)/\mu-\lambda-3/2} J_{(1+\lambda-\mu)/\mu}^{1/\mu}\left[(\tfrac{1}{2}x^{2}y^{2})^{1/\mu}\right] f(x)\,dx.$$

In particular, when $\mu = 1$, the above integral equation reduces to the Hankel integral equation. We shall call $f(x)$ the generalised Hankel transform of $g(y)$ in (1).

The object of the present note is to find two more inversion formulae for (1) in which the reciprocal kernels are not functions of the type $J_{\lambda}^{\mu}(x)$ but are different. In proving these inversion formulae we shall use some known results which, for the sake of elegance, I cite in §2.

2. Gupta (1948) has proved that

$$\int_{0}^{\infty} J_{\lambda}^{\mu}(sw^{\mu}) \sum_{r=0}^{\infty} \frac{\Gamma(l+r+k+\frac{1}{2}+m)\Gamma(l+r+k+\frac{1}{2}-m)}{r\,!\,\Gamma(1+l+r)\Gamma\{(1+\lambda+r)/\mu\}} (-)^{r} s^{-1+(\lambda+1+r)/\mu}\,ds$$
$$= w^{-\lambda-1+l+k} e^{-\frac{1}{2}w} W_{km}(w) \tag{2.1}$$

valid for $\frac{1}{2} \leq \mu \leq 1$, $\mathrm{Re}\lambda > -1$ and in the case $\mu = 1$ an additional condition $\mathrm{Re}(4k+4l-2\lambda \pm 4m+1) > 0$. Also

$$\int_{0}^{\infty} J_{\lambda}^{\mu}(sw) \sum_{r=0}^{\infty} \frac{\Gamma(1+\lambda+\mu\zeta+\mu r)(-)^{r}}{r\,!} s^{-r-\rho-1}\,ds = 2w^{\frac{1}{2}\rho} K_{\rho}(2w^{\frac{1}{2}}) \tag{2.2}$$

provided $0 < \mu \leq 1$, $\mathrm{Re}\,\lambda > -1$ and $\mathrm{Re}(4\rho+2\lambda+3) > 0$ in the case when $\mu = 1$.

Meijer (1933) has proved the following inversion formula for his generalisation of the Laplace transform:

If

$$f(t) = \int_{0}^{\infty} (st)^{-k-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k+\frac{1}{2},m}(st) g(s)\,ds$$

then

$$g(s) = \lim_{\lambda\to\infty} \frac{\Gamma(1-k+m)}{\Gamma(1-2m)} \cdot \frac{1}{2\pi i} \int_{\beta-\lambda i}^{\beta+\lambda i} e^{\frac{1}{2}st} M_{k-\frac{1}{2},m}(st)(st)^{-k-\frac{1}{2}} f(t)\,dt$$

provided that

(i) $\int_0^\infty e^{-\beta s} |g(s)|\,ds$ is convergent for $a \geq 0$, $\beta > a$,

(ii) $g(s)$ is defined for positive s and is integrable in the Riemann sense in every interval $0 < T_1 \leq t \leq T_2$,

(iii) $g(s)$ is a continuous function of s,

(iv) $k \leq m \leq -k$, $\beta > a$.

The last result that we shall need is due to Boas (1942), who has proved the following result:

Define the operators W and Q_k as

$$W[g(z)] = z^{2\nu-1}[z^{1-2\nu} g'(z)]'$$

and

$$Q_k[f(u)] = \frac{1}{(2k)!}(2k/\mu)^{2k+\frac{1}{2}-\nu} W^k[z^{\nu-\frac{1}{2}} f(z)]_{z=2k/u}$$

where the dashes denote differentiation with respect to z.

If

$$f(z) = (2z/\pi)^{\frac{1}{2}} \int_0^\infty t^{\frac{1}{2}} K_\nu(2t)\varphi(t)\,dt$$

is convergent and if $\varphi(t)$ is integrable in every finite interval of t, then for almost all t

$$\varphi(t) = \lim_{k\to\infty} Q_k[f(u)]$$

when $-\frac{1}{2} < \mathrm{Re}\gamma < \frac{1}{2}$.

3. The first inversion formula:

Let us consider the integral equation (1). Multiplying both sides of this by

$$\psi(x, t) \equiv 2^{\lambda+1}/\mu\, x^{2(\lambda+1)/\mu-\lambda-3/2}\, t^{3\mu/4+\frac{1}{2}\mu\lambda-\lambda-2} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(\lambda-k+m-\frac{3}{4}\mu-\frac{1}{2}\mu\lambda+2+r)\Gamma(\lambda-k-m-\frac{3}{4}\mu-\frac{1}{2}\mu\lambda+2+r)}{r!\,\Gamma(\lambda-2k-\frac{3}{4}\mu-\frac{1}{2}\mu\lambda+2+r)\Gamma\{(1+\lambda+r)/\mu\}} (-)^r [x/2t^{\frac{1}{2}\mu}]^{2r/\mu}.$$

and integrating from 0 to ∞ with respect to x, we have

$$\int_0^\infty f(x)\psi(x, t)\,dx = (1/2)^\lambda \int_0^\infty \psi(x, t)\,dx \int_0^\infty (xy)^{\lambda+\frac{1}{2}} J_\lambda^\mu(\tfrac{1}{4}x^2y^2) g(y)\,dy. \qquad (3.1)$$

If the change in the order of integration is permissible we get

$$\chi(t) \equiv \int_0^\infty f(x)\psi(x, t)\,dx = (\tfrac{1}{2})^\lambda \int_0^\infty y^{\lambda+\frac{1}{2}} g(y)\,dy \int_0^\infty x^{\lambda+\frac{1}{2}} J_\lambda^\mu(\tfrac{1}{4}x^2y^2)\psi(x, t)\,dx.$$

Integrating the inner integral with help of (2.1) we get, after a slight change in the variable,

$$\chi(t) = \int_0^\infty (st)^{-k-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k+\frac{1}{2},m}(st)\, g(s^{\frac{1}{2}\mu})\,ds \qquad (3.2)$$

valid for $\frac{1}{2} \leq \mu \leq 1$, $\operatorname{Re}\lambda > -1$ and in the case $\mu = 1$ with the additional condition $\operatorname{Re}(3\mu + 2\mu\lambda - 2\lambda + 4k \pm 4m - 5) < 0$.

Now applying the result (2.3) to (3.2) we have got the required inversion formula

$$g(s^{\frac{1}{2}\mu}) = \lim_{\eta \to \infty} \frac{1}{2\pi i} \frac{\Gamma(1-k+m)}{\Gamma(1+2m)} \int_{\beta-\eta i}^{\beta+\eta i} e^{\frac{1}{2}st}(st)^{-k-\frac{1}{2}} M_{k-\frac{1}{2},m}(st)\chi(t)dt$$

where $\chi(t)$ is defined above. This gives

$$g(s) = \lim_{\eta \to \infty} \frac{1}{2\pi i} \frac{\Gamma(1-k+m)}{\Gamma(1+2m)} \int_{\beta-\eta i}^{\beta+\eta i} \exp(\tfrac{1}{2}ts^{2/\mu}) M_{k-\frac{1}{2},m}(ts^{2/\mu})(ts^{2/\mu})^{\lambda-\frac{1}{2}}\chi(t)dt \qquad (3.3)$$

provided that $g(s^{\frac{1}{2}\mu})$ satisfies the conditions of Meijer's inversion formula mentioned in §2 and $\chi(t)$ exists.

It now only remains to justify the change in the order of integration in (3.1). To do so we see that (Wright, 1935)

$$\psi(x,\ t) \sim x^{\frac{1}{2}(5+2\lambda-3\mu-2\mu\lambda)} t^{\frac{1}{4}(3\mu^2-5\mu+2\mu^2\lambda-2\mu\lambda-4)} \exp(-kx^2/t^{\mu})$$

for $k > 0$ and $|x|$ large, and

$$\psi(x,\ t) = O[x^{(1/2\mu)(4\lambda+4-2\mu\lambda-3\mu)} t^{\frac{1}{4}(3\mu+2\mu\lambda-4\lambda-8)}]$$

for $|x|$ small.

The change in the order of integration is now valid provided

$$g(y) = O(y^{\xi}),\ \operatorname{Re}\xi < -1 \text{ for } |y| \text{ large}$$

and

$$= O(y^{\alpha}),\ \operatorname{Re}\alpha, > -1 \text{ for } |y| \text{ small},$$

with $\frac{1}{2} \leq \mu \leq 1$, $\operatorname{Re}(\alpha + \lambda + \frac{3}{2}) > 0$ and $\operatorname{Re}\lambda > -1$.

Thus our inversion formula (3.3) will be valid under the conditions

(i) $\operatorname{Re}\lambda > -1$; $\frac{1}{2} \leq \mu \leq 1$ and in the case $\mu = 1$ with an additional condition $\operatorname{Re}(3\mu + 2\mu\lambda - 2\lambda + 4k \pm 4m - 5) < 0$;

(ii)
$$g(y) = O(y^{\xi}),\ \operatorname{Re}\xi < -1 \text{ for } |y| \to \infty$$
$$= O(y^{\alpha}),\ \operatorname{Re}\alpha > -1 \text{ for } |y| \to 0$$

where $\operatorname{Re}(\alpha + \lambda + \frac{3}{2}) > 0$,

(iii) $g(y^{\frac{1}{2}\mu})$ is a continuous function of $y^{\frac{1}{2}\mu}$ and is defined for $y > 0$ and is integrable in the Riemann sense in every-interval $0 \leq T_1 \leq t \leq T_2$,

(iv) $\chi(t)$ exists.

4. The second inversion formula:

As in §3, let us next multiply both sides of (1) by [Gupta (1948)].

$$\mu^{-1}p^{2\nu}x^{-2\nu-\lambda-3/2}G^{\mu}_{1+\lambda+\mu\nu}(p^2/x^2).\dagger$$

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† where $G^{\mu}_{\lambda}(x) = \mu \sum_{0}^{\infty} \frac{\Gamma(\lambda+\mu r)(-x)^r}{r!}$, $\mu < 1$ is the generalised parabolic cylinder function given by Gupta In particular

$$G_{\lambda}^{\frac{1}{2}}(x = 2^{-\lambda}\Gamma(2\lambda)D_{-2\lambda}(x/\sqrt{2})e^{\frac{1}{4}x^2}.$$

Integrating both sides with respect to x from zero to infinity we get

$$\mu^{-1}p^{2\nu}\int_0^\infty f(x)x^{-2\nu-\lambda-3/2}G^{\mu}_{1+\lambda+\mu\nu}(p^2/x^2)dx$$

$$= (\tfrac{1}{2})^{\lambda}\mu^{-1}p^{2\nu}\int_0^\infty G^{\mu}_{1+\lambda+\mu\nu}(p^2/x^2)x^{-2\nu-1}dx\int_0^\infty y^{\lambda+\frac{1}{2}}J^{\mu}_{\lambda}(\tfrac{1}{4}x^2y^2)g(y)dy \qquad (4.1)$$

Changing the order of integration, if permissible, and integrating the x-integral on the right with the help of (2.2) we get for $0 < \mu \leq 1$, $\mathrm{Re}\lambda > -1$ and $\mathrm{Re}(4\nu+2\lambda+3) > 0$ that

$$\frac{2^{\lambda+\nu+\frac{1}{2}}}{\pi^{\frac{1}{2}}}\mu^{-1}p^{\nu+\frac{1}{2}}\int_0^\infty f(x)x^{-2\nu-\lambda-3/2}G^{\mu}_{1+\lambda+\mu\nu}(p^2/x^2)dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty (yp)^{\frac{1}{2}}K_\nu(yp).y^{\lambda+\nu}g(y)dy. \qquad (4.2)$$

Now, using the inversion formula (2.4) we get, for almost all y

$$y^{\nu+\lambda}g(y) = \lim_{k\to\infty} Q_k[\psi(p)] \qquad (4.3)$$

where

$$\psi(p) = \frac{2^{\lambda+\nu+\frac{1}{2}}}{\pi^{\frac{1}{2}}}\mu^{-1}p^{\nu+\frac{1}{2}}\int_0^\infty f(x)x^{-2\nu-\lambda-3/2}G^{\mu}_{1+\lambda+\mu\nu}(p^2/x^2)dx$$

and Q_k is the operator defined in (2.4). The above result holds provided that

(i) the integral on the right of (4.2) is absolutely convergent,

(ii) $\psi(p)$ exists and $y^{\nu+\lambda}g(y)$ is integrable in any finite interval of t,

(iii) $-\frac{1}{2} < \mathrm{Re}\,\nu < \frac{1}{2}$.

In order to justify the change in the order of integration we see on studying the behaviour of $G^{\mu}_{\lambda}(x)$ for large and small values of $|x|$, that for $0 < \mu \leq 1$

$$G^{\mu}_{\lambda}(x) = O(x^{-\lambda/\mu}) \quad \text{for } |x| \to \infty \text{ if } \lambda/\mu \neq \text{an integer}$$
$$= O(1) \quad \text{for } |x| \to 0.$$

Hence, the change in the order of integration will be permissible by absolute convergence if

(i) $\mathrm{Re}\,\lambda > -1$,

with

(ii) $$g(x) = O(x^{\beta}), \text{ for } |x| \to \infty$$
$$= O(x^{\alpha}), \text{ for } |x| \to 0$$

where $\mathrm{Re}(2\nu+\alpha+\lambda+\frac{3}{2}) > 0$.

Thus our second inversion formula (4.3) is valid under the conditions

(i) $-\frac{1}{2} < \mathrm{Re}\,\nu < \frac{1}{2}$, $\mathrm{Re}\,\lambda > -1$, $0 < \mu \leq 1$ and $\mathrm{Re}\,(4\nu+2\lambda+3) > 0$,

(ii) $y^{\lambda+\nu}g(y)$ is integrable in any finite internal of t and

$$g(y) = O(y^{\beta}), \text{ for } |y| \to \infty$$
$$= O(y^{\alpha}), \text{ for } |y| \to 0$$

where $\mathrm{Re}(2\nu+\alpha+\lambda+\frac{3}{2})>0$,

(iii) $\psi(p)$ exists.

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ANHARMONIC OSCILLATIONS OF A PARTICULAR MODEL

By

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Summary

The anharmonic radial oscillations of a stellar model consisting of a point mass, equal to one third of the total mass of the star, at the centre and the rest homogeneously distributed throughout the star, have been considered. The skewness and the period increase is calculated. Also the effects of the variation of the semi-amplitude and the variation of the effective ratio of specific heats on the skewness and the period are studied.

1. The theory of anharmonic oscillations was put forward by Rosseland (1943) and was well discussed by Bhatnagar and Kothari (1944). These oscillations for different models were studied only during the last three years. Schwarzschild (1949), C. Prasad (1949) and L. D. Chatterji (1951) investigated most of the models considering the effects of the higher modes also. The author (1951) studied the Roche-model (having an extreme concentration of mass) and also considered (1951) the effect of increasing semi-amplitude on the period and skewness for that model. In the present note the author has investigated a model for which one third of the whole mass of the star is concentrated in a point at the centre and the remaining two thirds is distributed throughout the star homogeneously. This gives a lesser central concentration of mass than the Roche-model.

2. The pulsation equation for the small adiabatic radial oscillations with finite amplitude is

$$\rho_0 r_0 \ddot{\eta} = -(1+\eta)^2 \frac{\partial}{\partial r_0}\left[P_0(1+\eta)^{-2\gamma}(1+\eta+r_0\eta')^{-\gamma}\right] - g_0\rho_0(1+\eta)^{-2}, \tag{1}$$

where P_0, ρ_0 and r_0 are the equilibrium values of pressure, density and the distance from the centre, γ the effective ratio of specific heats, η the relative displacement $\delta r/r_0$, and dashes denote differentiation with respect to r_0 and dots with respect to time.

For infinitesimally small oscillations the above equation can be solved (1948) giving a set of harmonic solutions in terms of a finite series, with periods $2\pi/\sigma_k$ and amplitudes $\eta_k{}'r_1)$. We express the present displacement η in terms of an infinite series of these characteristic functions and unknown time coefficients. We shall take these time factors as known in terms of a Fourier series and the Fourier coefficients will ultimately be determined.

Thus we have

$$\eta = \sum_k \eta_k(r_0)q_k(t). \tag{2}$$

Normalising η_k's to unity at the surface of the star we get the relative displacement at the surface as

$$\eta_R = \sum q_k(t). \tag{3}$$

We substitute this value of η from (2) in (1) and expand the right side in powers of q_k's neglecting those greater than two. Due to the orthogonal property of η_k's if we multiply the resulting equation by $r_0{}^3\eta_k$ and integrate for the whole star we get

$$\frac{d^2 q_k}{d\tau^2} + \beta_k q_k = \frac{\gamma}{\sigma_1{}^2 I_k}\left[\sum D_{iik} q_i{}^2 + 2\sum_{ij} D_{ijk} q_i q_j\right], \tag{4}$$

where

$$\tau = \sigma_1 t, \tag{5}$$

$$\beta_k = \sigma_k{}^2/\sigma_1{}^2, \tag{6}$$

$$I_k = \int_0^R \rho_0 r_0{}^4 \eta_k{}^2\, dr_0, \tag{7}$$

and

$$D_{ijk} = -\frac{1}{2}\left(3 - \frac{4}{\gamma}\right)(3\gamma+1)\int_0^R P_0' r_0^3 \eta_i \eta_j \eta_k\, dr_0 + \tfrac{1}{2}(3\gamma-1)\int_0^R P_0 r_0^4 (\eta_i \eta_j' \eta_k'$$
$$+ \eta_i' \eta_j \eta_k' + \eta_i' \eta_j' \eta_k)\, dr_0 + \tfrac{1}{2}(\gamma+1)\int_0^R P_0 r_0^5 \eta_i' \eta_j' \eta_k'\, dr_0. \tag{8}$$

3. Putting $x = r_0/R$ we have for this model

$$M(r_0) = \frac{2\pi}{3}\rho_0 R^3(1+2x^3), \tag{9}$$

$$g_0 = \frac{2\pi}{3} G\rho_0 R\left(\frac{1}{x^2} + 2x\right), \tag{10}$$

$$P_0 = \frac{2\pi}{3} G\rho_0{}^2 R^2\left(\frac{1}{x} - x^2\right). \tag{11}$$

Normalised η_1 and η_2 are

$$\left.\begin{aligned} \eta_1 &= x^q \\ \eta_2 &= -\frac{(q+3)(q+5)-\alpha}{9}\, x^q\left\{1 - \frac{6(q+4)x^3}{(q+3)(q+5)-\alpha}\right\} \end{aligned}\right\} \tag{12}$$

where

$$q = \sqrt{(1+\alpha)} - 1 \quad \text{and} \quad \alpha = 3 - \frac{4}{\gamma}.$$

Substituting these values of P_0 and g_0 from (10) and (11) in (4) we get

$$\ddot{q}_k + \beta_k q_k = \frac{1}{F_1 I_k}\left[\sum_i D_{iik} q_i^2 + 2\sum_{ij} D_{ijk} q_i q_j\right], \tag{13}$$

where

$$I_k = \int_0^1 x^4 \eta_k^2\, dx, \tag{14}$$

$$D_{ijk} = \frac{1}{2}\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\int_0^1 (1+2x^3)x\eta_i\eta_j\eta_k dx + \tfrac{1}{2}(3\gamma-1)\int_0^1 (1-x^3)x^3(\eta_i\eta_j'\eta_k'$$

$$+\eta_i'\eta_k\eta_j' + \eta_i'\eta_j'\eta_k)dx + \tfrac{1}{2}(\gamma+1)\int_0^1 (1-x^3)x^4\eta_i'\eta_j'\eta_k' dx, \qquad (15)$$

and

$$F_1 = q(q+5)+2\alpha. \qquad (16)$$

Hence we get

$$\left.\begin{aligned} \beta &= 1 \\ \beta_2 &= \frac{(q+3)(q+8)+2\alpha}{q(q+5)+2\alpha} \end{aligned}\right\}, \qquad (17)$$

$$I_1 = \frac{1}{2q+5}, \qquad (18)$$

$$I_2 = \left(\frac{(q+3)(q+5)-\alpha}{9}\right)^2 \left\{\frac{1}{2q+5} - \frac{6}{(q+3)(q+5)-\alpha} + \frac{36(q+4)^2}{\{(q+3)(q+5)-\alpha\}^2(2q+11)}\right\}. \qquad (19)$$

4. For $\alpha = 0\cdot 6$ we evaluate the coefficients. Consequently equations (13) reduce to :

$$\left.\begin{aligned} \frac{d^2q_1}{d\tau^2}+q_1 &= 2\cdot 8727q_1^2 - 0\cdot 312734q_1q_2 \\ \frac{d^2q_2}{d\tau^2}+10\cdot 8623q_2 &= -0\cdot 326024q_1^2 + 48\cdot 1602q_1q_2. \end{aligned}\right\} \qquad (20)$$

Here we substitute

$$\left.\begin{aligned} q_1 &= \sum_0^\infty a_r \cos(r\,n\tau) \\ q_2 &= \sum_0^\infty b_r \cos(r.n\tau) \end{aligned}\right\}, \qquad (21)$$

where $2\pi/n$ is the period of oscillation.

Equating the coefficients of $\cos(kn\tau)$ on both sides of (20) we get infinite number of infinite equations. We solve them approximately as explained in one of the author's previous papers (1951) choosing one arbitrary coefficient $a_1 = 0\cdot 05$ which gives the relative mean surface amplitude of oscillation nearly 1/20 of the radius of the star.

The solutions of (20) are as follows

$$\left.\begin{aligned} q_1 &= 0\cdot 003648 + 0\cdot 05\cos n\tau - 0\cdot 001218\cos 2n\tau + 0\cdot 000022\cos 3n\tau\ldots \\ q_2 &= -0\cdot 000042 - 0\cdot 000029\cos n\tau - 0\cdot 000069\cos 2n\tau - 0\cdot 000034\cos 3n\tau\ldots \end{aligned}\right\} \qquad (22)$$

where

$$n^2 = 0\cdot 98252. \qquad (23)$$

Thus we see that by the effect of the first mode the amplitude of the fundamental is decreased at the surface. The total surface displacement for the star is

$$\eta_R = 0{\cdot}003006 + 0{\cdot}049971 \cos n\tau - 0{\cdot}001287 \cos 2n\tau - 0{\cdot}0000113 \cos 3n\tau \ldots \tag{24}$$

The skewness for the time-velocity curve is found to be 1.133 for this model and the period increase is 0·89%. The table 1 shows the values for Roche-model and the present one for comparision, where x_1 and x_2 denote the external and internal amplitudes respectively and semi-amplitude is the numerical mean of x_1 and x_2.

TABLE 1

	x_1	x_2	Semi-amplitude $=\frac{1}{2}(x_1+x_2)$	Skewness	Period increase $=100\delta P/P_0$
Present model	0·0523	0·0476	0·04996	1·133	0·89
Roche-model	0·0569	0·0444	0·0507	1·086	1·08

From the above table it appears that the concentration of mass at the centre of the star has little effect on the skewness and the period increase. The effect on the two quantities is in the opposite directions, *i.e.* if one increases the other decreases. The semi-amplitude in both the cases remains almost the same, although the external and internal amplitudes differ.

5. Now we calculate the coefficients in (13) for $\alpha = 0{\cdot}4$ and $\alpha = 0{\cdot}2$ and compare the results to study the effect of the change of adiabatic exponent on the skewness and the period.

(*i*) For $\alpha = 0{\cdot}4$ equations (13) become

$$\left.\begin{aligned} \frac{d^2q_1}{d\tau^2} + q_1 &= 2{\cdot}71601 q_1^2 - 0{\cdot}406354 q_1 q_2 \\ \frac{d^2q_2}{d\tau^2} + 15{\cdot}346 q_2 &= -0{\cdot}430344 q_1^2 + 60{\cdot}507 q_1 q_2 \end{aligned}\right\} \tag{25}$$

Solutions of this are

$$\left.\begin{aligned} q_1 &= 0{\cdot}003428 + 0{\cdot}05 \cos n\tau - 0{\cdot}0011495 \cos 2n\tau + 0{\cdot}0000198 \cos 3n\tau \ldots \\ q_2 &= -0{\cdot}000038 - 0{\cdot}000022 \cos n\tau - 0{\cdot}0000574 \cos 2n\tau - 0{\cdot}0000083 \cos 3n\tau \ldots \end{aligned}\right\} \tag{26}$$

where

$$n^2 = 0{\cdot}98448 \tag{27}$$

The displacement at the surface of the star is

$$\eta_R = 0{\cdot}00339 + 0{\cdot}049978 \cos n\tau - 0{\cdot}001201 \cos 2n\tau + 0{\cdot}0000115 \cos 3n\tau \ldots. \tag{28}$$

(ii) For $\alpha = 0{\cdot}2$ the equations (13) take the form

$$\left.\begin{aligned} \frac{d^2q_1}{d\tau^2} + q_1 &= 2{\cdot}59187q_1^2 - 0{\cdot}535991q_1q_2 \\ \frac{d^2q_2}{d\tau^2} + 28{\cdot}723q_2 &= -0{\cdot}577763q_1^2 + 99{\cdot}7566q_1q_2 \end{aligned}\right\} \tag{29}$$

Solving these equations we get

$$\left.\begin{aligned} q_1 &= 0{\cdot}003269 + 0{\cdot}05\cos n\tau - 0{\cdot}001095\cos 2n\tau + 0{\cdot}000018\cos 3n\tau \ldots \\ q_2 &= -0{\cdot}000027 - 0{\cdot}000013\cos n\tau - 0{\cdot}000031\cos 2n\tau - 0{\cdot}0000023\cos 3n\tau \ldots \end{aligned}\right\} \tag{30}$$

where

$$n^2 = 0{\cdot}98587. \tag{31}$$

The displacement for the stellar surface is given by

$$\eta_R = 0{\cdot}003242 + 0{\cdot}0499866\cos n\tau - 0{\cdot}001126\cos 2n\tau + 0{\cdot}000016\cos 3n\tau. \tag{32}$$

The values for skewness and period for these two cases are also obtained from the respective time-velocity curves. The results are collected in Table 2 for comparative study. From there it seems that by increasing the value of γ the period and the skewness both increase although the increments are hardly appreciable. Hence obviously by varying the value of γ between the physically possible limits the skewness and the period cannot be brought in agreement with the observed values.

TABLE 2

α	x_1	x_2	Semi-amplitude $=\frac{1}{2}(x_1+x_2)$	skewness	Period increase $100\delta P/P_0$
0·6	0·05228	0·04764	0·04996	1·133	0·89
0·4	0·05218	0·04780	0·04999	1·124	0·79
0·2	0·05212	0·04789	0·05000	1·118	0·71

6. Now we shall vary the semi-amplitude keeping $\gamma = \frac{5}{3}$ and study the effect of this variation on the skewness and period. The range of variation is chosen from $\frac{1}{100}R$ to $\frac{1}{10}R$, R being the radius of the star. For the higher values our approximations become very crude. The results are collected in Table 3, in which the corresponding values for the Roche-model have also been given from the previous paper of the uthor (1951).

TABLE 3

a_1	Semi-amplitudes $= \frac{x_1+x_2}{2}$		Skewness		Period-increase $=100\delta P/P_0$	
	Present model	Roche-model	Present model	Roche-model	Present model	Roche-model
0·01	0·01000	0·010006	1·028	1·017	0·035	0·043
0·02	0·019997	0·02005	1·05	1·034	0·138	0·171
0·03	0·029993	0·0302	1·08	1·048	0·311	0·387
0·04	0·039984	0·0404	1·108	1·063	0·555	0·691
0·05	0·049967	0·0507	1·133	1·086	0·872	1·085
0·06	0·059964	0·0613	1·16	1·111	1·263	1·576
0·07	0·06991	0·072	1·195	1·13	1·781	2·153
0·08	0·079866	0·083	1·23	1·155	2·279	2·853
0·09	0·08981	0·0943	1·26	1·182	2·912	3·654
0·1	0·09974	0·106	1·284	1·20	3·612	4·57

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STRESSES IN AN INFINITE STRIP DUE TO A NUCLEUS OF THERMO-ELASTIC STRAIN INSIDE IT

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Introduction—The stresses in an infinite strip of a thin plate of isotropic elastic material due to various distribution of loads have been found by Filon (1903), Melan (1925), Howland (1929) and others. The *direct* method of stress determination given by Sen (1938) has been utilized by the same author (1942) to find the stresses when an isolated couple acts at a point midway between the edges of the infinite strip. In the present paper, a nucleus of Thermo-elastic strain, instead of an isolated couple, is assumed to exist ; the nucleus is such as would be produced by heating to a certain temperature an element of area of the plate while the remaining part of the plate is kept at zero temparature.

Solution of the Problem.—Let the axes of x and y be in a plane parallel to the faces of the strip of which the edges are given by $x = \pm a$. If an element of area $d\Omega$ surrounding the origin be heated to the temperature T in the manner as described above, the average stressesin a thin plate evtending in all directions are given by, (Goodier, 1937, Sen, 1950).

$$\left.\begin{aligned} \widehat{xx}_1 &= \frac{E\alpha T d\Omega}{2\pi}\,\frac{y^2-x^2}{(x^2+y^2)^2} \\ \widehat{yy}_1 &= \frac{E\alpha T d\Omega}{2\pi}\,\frac{x^2-y^2}{(x^2-y^2)^2} \\ \widehat{xy}_1 &= -\frac{E\alpha T d\Omega}{\pi}\,\frac{xy}{(x^2+y^2)^2} \end{aligned}\right\} \tag{1}$$

where E is the Young's modulus and α, the coefficient of linear expansion.

Putting

$$Q = \frac{E\alpha T d\Omega}{2\pi} \tag{2}$$

we have, on the edge $x = a$

$$\left.\begin{aligned} (\widehat{xx}_1)_{x=a} &= Q\frac{y^2-a^2}{(a^2+y^2)^2} \\ (\widehat{xy}_1)_{x=a} &= -2Q\frac{ay}{(a^2+y^2)^2} \end{aligned}\right\} \tag{3}$$

and on the edge $x = -a$

$$\left.\begin{aligned} (\widehat{xx}_1)_{x=-a} &= Q\frac{y^2-a^2}{(a^2+y^2)^2} \\ (\widehat{xy}_1)_{x=-a} &= 2Q\frac{ay}{(a^2+y^2)^2} \end{aligned}\right\} \tag{4}$$

To nullify the stresses (3) and (4) on the edges, we are to superpose a stress system $\widehat{xx}_2$, $\widehat{yy}_2$, $\widehat{xy}_2$ such that on the edge $x = a$

$$\left.\begin{aligned}(\widehat{xx}_2)_{x=a} &= -Q\frac{y^2-a^2}{(a^2+y^2)^2} = f_1(y)\\ (\widehat{xy}_2)_{x=a} &= 2Q\frac{ay}{(a^2+y^2)^2} = f_2(y)\end{aligned}\right\}\tag{5}$$

and on the edge $x = -a$

$$\left.\begin{aligned}(\widehat{xx}_2)_{x=-a} &= f_1(y)\\ (\widehat{xy}_2)_{x=-a} &= -f_2(y)\end{aligned}\right\}\tag{6}$$

The superposed stresses must satisfy the equations of equilibrium and the consistency relations, These are given by, (Sen, 1938)

$$\left.\begin{aligned}\widehat{xx}_2 &= -\tfrac{1}{2}x\frac{\partial \odot_2}{dx} + \varphi_1(x,y)\\ \widehat{yy}_2 &= \odot_2 + \tfrac{1}{2}x\frac{\partial \odot_2}{\partial x} - \varphi_1(x,y)\\ \widehat{xy}_2 &= -\tfrac{1}{2}x\frac{\partial \odot_2}{\partial y} + \varphi_2(x,y)\end{aligned}\right\}\tag{7}$$

where $\odot_2 \equiv \widehat{xx}_2 + \widehat{xy}_2$ is a plane harmonic function and $\varphi_1(x, y)$ and $\varphi_2(x,y)$ are also plane harmonic functions satisfying the relations

$$\left.\begin{aligned}\tfrac{1}{2}\frac{\partial \odot_2}{\partial x} &= \frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\\ \tfrac{1}{2}\frac{\partial \odot_2}{\partial y} &= \frac{\partial \varphi_1}{\partial y} + \frac{\partial \varphi_2}{\partial x}\end{aligned}\right\}\tag{8}$$

We assume that

$$\left.\begin{aligned}\varphi_1(x,y) &= \int_0^\infty A_m \cosh mx \cos my \, dm\\ \varphi_1(x,y) &= \int_0^\infty B_m \sinh mx \sin my \, dm\end{aligned}\right\}\tag{9}$$

where A_m and B_m are function of m.

Substituting the values of φ_1 and φ_2 in the relations (8) we get

$$\left.\begin{aligned}\tfrac{1}{2}\frac{\partial \odot_2}{\partial x} &= \int_0^\infty m(A_m + B_m)\sinh mx \cos my \, dm\\ \tfrac{1}{2}\frac{\partial \odot_2}{\partial y} &= -\int_0^\infty m(A_m + B_m)\cosh mx \sin my \, dm\end{aligned}\right\}\tag{10}$$

from which we have

$$\odot_2 = 2\int_0^\infty (A_m + B_m) \cosh mx \cos my \, dm \tag{11}$$

Substituting the values of $\odot_2$, φ_1 and φ_2 in we get

$$\left.\begin{aligned} \widehat{xx}_2 &= \int_0^\infty [A_m(\cosh mx - mx \sinh mx) - B_m mx \sinh mx]\cos my \, dm\,] \\ \widehat{xy}_2 &= \int_0^\infty [A_m mx \cosh mx + B_m(mx \cosh mx) + \sinh mx)] \sin my \, dm \end{aligned}\right\} \tag{12}$$

From the boundary conditions (5) and (6), we have, by the help of relations (12).

$$\left.\begin{aligned} (2/\pi)^{\frac{1}{2}} \int_0^\infty F_1(m) \cos my \, dm &= f_1(y) \\ (2/\pi)^{\frac{1}{2}} \int_0^\infty F_2(m) \sin my \, dm &= f_2(y) \end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned} (2/\pi)^{\frac{1}{2}} F_1(m) &= A_m (\cosh ma - ma \sinh ma) - B_m ma \sinh ma \\ (2/\pi)^{\frac{1}{2}} F_2(m) &= A_m ma \cosh ma + B_m(ma \cosh ma + \sinh ma) \end{aligned}\right\} \tag{14}$$

By the 'transform theorems', we have from (13),

$$\left.\begin{aligned} F_1(m) &= (2/\pi)^{\frac{1}{2}} \int_0^\infty f_1(y) \cos my \, dy = -Q(2/\pi)^{\frac{1}{2}} \int_0^\infty \frac{y^2 - a^2}{(a^2 + y^2)^2} \cos my \, dy \\ F_2(m) &= (2/\pi)^{\frac{1}{2}} \int_0^\infty f_2(y) \sin my \, dy = 2Q a \int_0^\infty \frac{y}{(a^2 + y^2)^2} \sin my \, dy \end{aligned}\right\} \tag{15}$$

Evaluating infinite integrals in (15) and substituting the values of $F_1(m)$ and $F_2(m)$ from (14), we get

$$\left.\begin{aligned} A_m (\cosh ma - ma \sinh ma) - B_m ma \sinh ma &= Qm e^{-ma} \\ A_m ma \cosh ma + B_m(ma \cosh ma + \sinh ma) &= Qm e^{-ma} \end{aligned}\right\} \tag{16}$$

These results are satisfied if

$$\left.\begin{aligned} A_m &= \frac{2Qm(ma + e^{-ma} \sinh ma)}{2ma + \sinh 2ma} \\ B_m &= \frac{2Qm(e^{-ma} \cosh ma - ma)}{2ma + \sinh 2ma} \end{aligned}\right\} \tag{17}$$

Thus, from (11) and (12)

$$\left.\begin{aligned} \odot_2 &= 4Q \int_0^\infty \frac{m \cosh mx \cos my}{2am + \sinh 2ma} dm \\ \widehat{xx}_2 &= 2Q \int_0^\infty \frac{m \cos my}{2am + \sinh 2ma} [(ma + e^{-ma} \sinh ma) \cosh ma - mx \sinh mx] dm \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned}\widehat{xy}_2 &= 2Q\int_0 \frac{m\sin my}{2ma+\sinh 2ma}[mx\cosh mx+(e^{-ma}\cosh ma-ma)\sinh mx]dm\\ \widehat{yy}_2 = \odot_2-\widehat{xx}_2 &= 2Q\int_0^\infty \frac{m\cos my}{2ma+\sinh 2ma}[2\cosh mx+mx\sinh mx-\cosh mx\\ &\qquad\qquad (ma+e^{-ma}\sinh ma)]dm\end{aligned}\right\} \quad (19)$$

The resultant stresses can now be written as

$$\left.\begin{aligned}\widehat{xx} &= \widehat{xx}_1+\widehat{xx}_2\\ \widehat{yy} &= \widehat{yy}_1+\widehat{yy}_2\\ \widehat{xy} &= \widehat{xy}_1+\widehat{xy}_2\\ \odot &= \widehat{xx}+\widehat{yy} = 4Q\int_0^\infty \frac{m\cosh mx\cos my}{2ma+\sinh 2ma}dm\end{aligned}\right\} \quad (20)$$

where the two component stress systems are given by (1) and (19)

Since $\widehat{xx}=0$ on the edge $x=a$, we have

$$(\widehat{yy})_{x=a} = (\odot)_{x=a} = 4Q\int_0^\infty \frac{m\cosh ma\cos my}{2am+\sinh 2ma}dm.$$

Putting $ma=u$ and $y=a\eta$, we get, on substituting the values of Q from (2).

$$(yy)_{x=a} = \frac{2E\alpha T d\Omega}{\pi a^2}\mathrm{I}(\eta) \quad (21)$$

where

$$\mathrm{I}(\eta) = \int_0^\infty \frac{u\cosh u\cos u\eta}{2u+\sinh 2u}du \quad (22)$$

When η is small,

$$\mathrm{I}(\eta) \sim \int_0^\infty \frac{u\cosh u}{2u+\sinh 2u}du-\frac{\eta^2}{2}\int_0^\infty \frac{u^3\cosh u}{2u+\sinh 2u}du. \quad (23)$$

The values of the infinite integrals, as given by Filon (1903), are

$$\int_0^\infty \frac{u\cosh u}{2u+\sinh 2u}du = 0{\cdot}918, \quad \int_0^\infty \frac{u^3\cosh u}{2u+\sinh 2u}du = 5{\cdot}750 \quad (24)$$

Hence, when η is small

$$\mathrm{I}(\eta) \sim 0{\cdot}918-2{\cdot}875\eta^2 \quad (25)$$

similarly, the value $\widehat{yy}$ on the other edge $x=-a$ can be calculated.

In conclusion, I offer my grateful thanks to Dr. B. Sen for his kind help in the preparation of this paper.

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ON THE EFFECT OF A SMALL SPHERICAL CAVITY IN A SEMI-INFINITE ELASTIC SOLID UNDER STRESSES PRODUCED BY A COUPLE ON THE PLANE BOUNDARY

By

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Introduction: Dipolar co-ordinates have recently been used by Sternberg and Sadowsky (1952) in solving some three dimensional problems of elasticity. In the problems discussed by the authors the symmetrical displacement in which the azimuthal component was absent was only considered. In the present paper the solution of a three dimensional problem involving the azimuthal component only has been obtained.

Dipolar co-ordinates have been used here to find the concentration of stresses due to a spherical cavity situated near the plane face of a semi-infinfte solid when the stresses have been produced by a couple acting at the plane surface with its axis perpendicular to it.

Nomenclature:

x, y, z = Cartesian co-ordinates.

ϱ, φ, z = Cylindrical co-ordinates.

ξ, η, φ = Dipolar co-ordinates.

a = A positive length.

h_1, h_2, h_3 = Stretch ratios.

r = Radius of the sphere.

d = Distance of the centre of the sphere from the plane surface.

u, v, w = Components of displacements in dipolar co-ordinates.

Δ = Dilatation.

$2\varpi_\xi, 2\varpi_\eta\ 2\varpi_\varphi$ = Components of rotation in dipolar co-ordinates.

$\sigma_\xi, \sigma_\eta, \sigma_\varphi$ = Normal stress-components in dipolar co-ordinates.

$\zeta_{\xi\eta}, \zeta_{\xi\varphi}, \zeta_{\eta\varphi}$ = Shearing stress-components in dipolar co-ordinates.

M = Moment of the couple.

A'_n, B'_n, A_n, B_n = Constants.

1. The Coordinates.

We take the curvilinear co-ordinates defined by

$$\xi + i\eta = \log \frac{\varrho + i(z+a)}{\varrho + i(z-a)} \qquad (1.1)$$

where $\varrho = (x^2+y^2)^{\frac{1}{2}}$, x, y, z the cartesian co-ordinates and a, a positive length.

If ϱ, φ, z be the cylindrical co-ordinates, we get

$$\left.\begin{aligned} x &= \frac{a \sin\eta \cos\varphi}{\cosh\xi - \cos\eta} \\ y &= \frac{a \sin\eta \sin\varphi}{\cosh\xi - \cos\eta} \\ \text{and} \quad z &= \frac{a \sinh\xi}{\cosh\xi - \cos\eta}. \end{aligned}\right\} \qquad (1.2)$$

The relations

$$\left.\begin{aligned} \frac{1}{h_1^2} &= \left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2 + \left(\frac{\partial z}{\partial\xi}\right)^2 \\ \frac{1}{h_2^2} &= \left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2 + \left(\frac{\partial z}{\partial\eta}\right)^2 \\ \text{and} \quad \frac{1}{h_3^2} &= \left(\frac{\partial u}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 + \left(\frac{\partial z}{\partial\varphi}\right)^2 \end{aligned}\right\} \qquad (1.3)$$

yield

$$\left.\begin{aligned} h_1 = h_2 &= \frac{1}{a}(\cosh\xi - \cos\eta) \\ \text{and} \quad h_3 = \frac{1}{\varrho} &= \frac{1}{a \sin\eta}(\cosh\xi - \cos\eta) \end{aligned}\right\}. \qquad (1.4)$$

The surfaces ξ = constants give a series of coaxial spheres having a common radical plane $\xi = 0$ (or $z = 0$). We take the plane surface to be given by $\xi = 0$ and ξ is taken positive towards the interior of the body. Now a sphere of radius r having its centre at a distance d from the plane along the z-axis can be obtained by properly adjusting a and α by the relations

$$\left.\begin{aligned} r &= a \operatorname{cosech} \alpha \\ \text{and} \quad d &= a \coth \alpha \end{aligned}\right\} \qquad (1.5)$$

$\xi = \alpha$, giving the spherical surface.

2. The equations of equilibrium and the required solutions.

Let u, v, w represent the displacements u_ξ, u_η, u_φ. Here we take u and v to be zero and w independant of φ.

Then

$$\left.\begin{aligned}\Delta &= 0 \ldots\ldots\ldots\ldots\ldots\\ 2\varpi_\xi &= h_2h_3\left\{\frac{\partial}{\partial\eta}\left(\frac{w}{h_3}\right)\right\}\ldots\ldots\\ 2\varpi_\eta &= h_3h_1\left\{-\frac{\partial}{\partial\xi}\left(\frac{w}{h_3}\right)\right\}\ldots\\ 2\varpi_\varphi &= 0 \ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \quad (2.1)$$

and

Two of the equations of equlibrium (Love, 1944) are identically satisfied and the third takes the form

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{\varrho}\frac{\partial}{\partial\xi}(w\varrho)\right\}+\frac{\partial}{\partial\eta}\left\{\frac{1}{\varrho}\frac{\partial}{\partial\eta}(w\varrho)\right\} = 0 \quad (2.2)$$

The solution of this equation can be written as

$$w\varrho = \frac{1}{(\cosh\xi-\cos\eta)^{\frac{1}{2}}}\sum_{n=0}^{\infty}\{A_n'\cosh(n+\tfrac{1}{2})\xi+B_n'\sinh(n+\tfrac{1}{2})\xi\}[P_{n-1}(\cos\eta)-P_{n+1}(\cos\eta)] \quad (2.3)$$

when P_n is a Legendre's function of the n-th degree, and A_n', B_n' are constants.

Since

$$(\mu_1^2-1)P_n'(\mu_1) = \frac{n(n+1)}{2n+1}(P_{n+1}(\mu_1)-P_{n-1}(\mu_1)) \quad (2.4)$$

we can take as a solution suitable for our problem

$$w_2 = \frac{(\cosh\xi-\cos\eta)^{\frac{1}{2}}\sin\eta}{a}\sum_{n=2}^{\infty}\{A_n\cosh(n+\tfrac{1}{2})\xi+B_n\sinh(n+\tfrac{1}{2})\xi\}P_n' \quad (2.5)$$

P_n' denoting differentiation with respect to $\cos\eta$. The stresses that do not vanish are given by

$$\left.\begin{aligned}(\tau_{\xi\varphi})_2 &= \mu\sin\eta\frac{\partial}{\partial\xi}\left(\frac{\omega_2}{\varrho}\right)\\ &= \frac{\mu\sin\eta}{a^2}\Big[\tfrac{3}{2}(\cosh\xi-\cos\eta)^{\frac{1}{2}}\sinh\xi\sum_{n=2}^{\infty}\{A_n\cosh(n+\tfrac{1}{2})\xi+B_n\sinh(n+\tfrac{1}{2})\xi\}P_n'\\ &\qquad+(\cosh\xi-\cos\eta)^{3/2}\sum_{n=2}^{\infty}(n+\tfrac{1}{2})\{A_n\sinh(n+\tfrac{1}{2})\xi+B_n\cosh(n+\tfrac{1}{2})\xi\}P_n'\Big]\\ &\text{and}\\ (\tau_{\eta\varphi})_2 &= \mu\sin\eta\frac{\partial}{\partial\eta}\left(\frac{\omega_2}{\varrho}\right)\\ &= \frac{\mu\sin^2\eta}{a^2}\Big[\tfrac{3}{2}(\cosh\xi-\cos\eta)^{1/2}\sum_{n=2}^{\infty}\{A_n\cosh(n+\tfrac{1}{2})\xi+B_n\sinh(n+\tfrac{1}{2})\xi\}P_n'\\ &\qquad-(\cosh\xi-\cos\eta)^{3/2}\sum_{n=2}^{\infty}\{A_n\cosh(n+\tfrac{1}{2})\xi+B_n\sinh(n+\tfrac{1}{2})\xi\}P_n''\end{aligned}\right\} \quad (2.6)$$

μ, being the rigidity of the solid and P_n'' is the second derivative of P_n with respect to $\cos\eta$.

If a couple of moment M with its axis normal to the plane surface act at the origin on this plane, we have the displacement due to it in the form (Love, *loc. cit.*)

$$u_1 = v_1 = 0$$

and

$$\omega_1 = -\frac{M}{4\pi\mu}\cdot\frac{\varrho}{(\varrho^2+z^2)^{3/2}}. \tag{2.7}$$

Alternatively, we can put

$$\omega_1 = \frac{M \sin\eta\,(\cosh\xi-\cos\eta)^{\frac{1}{2}}}{4\pi\mu a^2}\,2\sqrt{2}\sum_{n=1}^{\infty}(-1)^n e^{-(n+\frac{1}{2})\xi}P_n' \tag{2.8}$$

From this result we get

$$\left.\begin{aligned}(\tau_{\xi\varphi})_1 &= \frac{M\mu\sin\eta}{\sqrt{2}\pi\mu a^3}\Big[(\cosh\xi-\cos\eta)^{3/2}\sum_{n=1}^{\infty}(-1)^n\{-(n+\tfrac{1}{2})e^{-(n+\frac{1}{2})\xi}P_n'\\ &\qquad\qquad +\tfrac{3}{2}(\cosh\xi-\cos\eta)^{\frac{1}{2}}\sinh\xi\sum_{n=1}^{\infty}(-1)^n e^{-(n+\frac{1}{2})\xi}P_n'\Big]\\ \text{and}\qquad&\\ (\tau_{\eta\varphi})_1 &= \frac{M\mu\sin^2\eta}{\sqrt{2}\pi\mu a^3}\Big[-(\cosh\xi-\cos\eta)^{3/2}\sum_{n=1}^{\infty}(-1)^n e^{-(n+\frac{1}{2})\xi}P_n''\\ &\qquad\qquad +\tfrac{3}{2}(\cosh\xi-\cos\eta)^{\frac{1}{2}}\sum_{n=1}^{\infty}(-1)^n e^{-(n+\frac{1}{2})\xi}P_n'\Big]\end{aligned}\right\} \tag{2.9}$$

Let the complete displacement be given by

$$w = w_1 + w_2 \tag{2.10}$$

Now w_2 is to be so chosen as to make

$$\left.\begin{aligned}(\tau_{\xi\varphi})_2 &= 0, \quad \text{on} \quad \xi = 0\\ (\tau_{\xi\varphi})_1 + (\tau_{\xi\varphi})_2 &= 0, \quad \text{on} \quad \xi = \alpha\end{aligned}\right\} \tag{2.11}$$

where suffixes 1 and 2 denote stresses calculated from ω_1 and ω_2.

On using the recurrence formulae

$$\left.\begin{aligned}P_{n+1}'(\mu_1) - \mu_1 P_n'(\mu_1) &= (n+1)P_n(\mu_1)\\ \text{and}\qquad \mu_1 P_n'(\mu_1) - P_{n-1}'(\mu_1) &= nP_n(\mu_1)\end{aligned}\right\} \tag{2.12}$$

we find that the above conditions are satisfied if we take

$$\left.\begin{aligned}B_n &= 0\\ A_2 &= \frac{M\sqrt{2}e^{-5\alpha/2}}{\pi\mu a\sinh\frac{5}{2}\alpha}\\ 4\sinh\tfrac{7}{2}\alpha . A_3 &= [3\sinh\alpha\cosh\tfrac{5}{2}\alpha + 5\cosh\alpha\sinh\tfrac{5}{2}\alpha]A_2\\ &\qquad -\frac{\sqrt{2}M}{\pi\mu a}(e^{-3\alpha/2}+4e^{-7\alpha/2})\\ \text{and when } n \geq 2 &\\ [(n+2)\sinh(n+\tfrac{3}{2})\alpha]A_{n+1} &= [3\sinh\alpha\cosh(n+\tfrac{1}{2})\alpha\\ &\quad +(2n+1)\cosh\alpha\sinh(n+\tfrac{1}{2})\alpha]A_n + (n-1)A_{n-1}\sinh(n-\tfrac{1}{2})\alpha\\ &\qquad -2M_n[(n-1)e^{-(n-\frac{1}{2})\alpha}+(n+2)e^{-(n+3/2)\alpha}]\end{aligned}\right\} \tag{2.13}$$

where

$$M_n = \frac{(-1)^n M}{\sqrt{2}.\pi\mu a}. \tag{2.14}$$

The complete stress at $\xi = \alpha$ is obtained from (2.5), (2.9) and (2.14) as follows:

$$\begin{aligned}(\tau_{\eta\varphi})_{\xi=\alpha} &= [(\tau_{\eta\varphi})_1 + (\tau_{\eta\varphi})_2] \\ &= \frac{\mu \sin^2\eta}{a^2}(\cosh\alpha - \cos\eta)^{\frac{1}{2}} \sum_{n=1}^{\infty} [A_n \cosh(n+\tfrac{1}{2})\alpha - M_n e^{-(n+\frac{1}{2})\alpha}] \\ &\qquad \times \{\tfrac{3}{2}P'_n - (\cosh\alpha - \cos\eta)P''_n\}. \end{aligned} \tag{2.15}$$

where $A_1 = 0$.

The series evidently converges.

When $\alpha = 2$, we obtain on retaining first few terms

$$\begin{aligned}\frac{\pi a^3}{M\sqrt{2}}\tau_{\eta\varphi} &= \sin^2\eta\,(3{\cdot}7622 - \cos\eta)^{\frac{1}{2}} \times [0{\cdot}08734 + 0{\cdot}0033(7{\cdot}5\cos\eta - 11{\cdot}2866) \\ &\qquad + 0{\cdot}0366(26{\cdot}25\cos^2\eta - 56{\cdot}433\cos\eta - 2{\cdot}25)] \end{aligned} \tag{2.16}$$

approximately.

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ON CERTAIN FORMULAE IN OPERATIONAL CALCULUS

By

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The object of the present note is to derive certain formulae in operational calculus and to obtain certain contour integrals for Bessel Functions and Weber's Parabolic Cylinder Functions.

1. We have the formulae:

$$\text{If } \varphi(p) \doteqdot f(t); \ \psi(p) \doteqdot g(t),$$

then

$$\int_0^\infty \frac{\varphi(t)g(t)}{t}\,dt = \int_0^\infty \frac{f(t)\psi(t)}{t}\,dt \quad \text{(Goldstein, 1932).} \tag{1}$$

Let

$$\psi(p) = \frac{p}{p+a}, \quad \text{then } g(t) = \exp\,(-at).$$

The relation (1) becomes

$$\int_0^\infty \frac{f(t)}{a+t}\,dt = \int_0^\infty \exp\,(-ax)\,\frac{\varphi(x)}{x}\,dx. \tag{2}$$

Let us write t^2 for t, a^2 for a in (2) and multiply both sides by $J_m(ab)a^{m+1}$. Integrating with respect to a between the limits zero and infinity, we have

$$2\int_0^\infty tf(t^2)dt\int_0^\infty \frac{J_m(ab)a^{m+1}}{a^2+t^2}\quad da = \int_0^\infty \frac{\varphi(x)}{x}\,dx\int_0^\infty \exp\,(-a^2x)J_m(ab)a^{m+1}da\,;$$

where we have assnmed that change in the order of integration is permissible.

Putting $b = p$ and $x = 1/v^2$, we have on integration

$$\int_0^\infty t^{m+1}f(t^2)K_m(pt)dt = \frac{p^m}{2^{m+1}}\int_0^\infty v^{2m+1}\exp\,(-\tfrac{1}{4}p^2v^2)\varphi(1/v^2)dv$$

$$= \frac{p^m}{2^{m+2}}\int_0^\infty v^m\exp\,(-\tfrac{1}{4}p^2v)\varphi(1/v)dv. \tag{2}$$

If we put $m = -\frac{1}{2}$, we fall back upon the result (Humbert, 1934, p. 12).

$$\int_0^\infty \exp\,(-pt)f(t^2)dt = \frac{1}{\pi^{\frac{1}{2}}}\int_0^\infty \exp\,(-\tfrac{1}{4}p^2v^2)\varphi(1/v^2)dv.$$

Again writing $4p$ for p^2, we find that

$$\frac{4}{p^{\frac{1}{2}m-1}}\int_0^\infty t^{m+1}f(t^2)K_m(2p^{\frac{1}{2}}t)dt \doteqdot t^m\varphi(1/t)\,;$$

So that if $\varphi(p) \doteqdot f(t)$, then

$$2p\int_0^\infty f(t)K_0(2\sqrt{(pt)})dt \doteqdot \varphi(1/t). \tag{4}$$

2. Let us now write $2p$ for p and $n-\frac{1}{2}$ for m in (8). We obtain

$$4\int_0^\infty z^{n+\frac12}f(z^2)K_{n-\frac12}(2pz)dz = p^{n-\frac12}\int_0^\infty v^{n-\frac12}\exp(-p^2v)\varphi(1/v)dv$$

Let us multiply both sides by $p^{n-\frac12}\exp(-\frac14 p^2)D_{2n}(p)$, and integrate between the limits zero and infinity. We have on changing the order of integration (assuming that is permissible)

$$4\int_0^\infty z^{n+\frac12}f(z^2)dz\int_0^\infty p^{n-\frac12}\exp(-\tfrac14 p^2)K_{n-\frac12}(2pz)D_{2n}(p)dp$$

$$=\int_0^\infty v^{n-\frac12}\varphi(1/v)dv\int_0^\infty p^{2n-1}\exp[-p^2(v+\tfrac14)]D_{2n}(p)dp$$

On using a result due to Mitra (1946, p. 171) the left side becomes

$$\pi^{\frac12}(-1)^n 2^{n+1}\Gamma(2n)\int_0^\infty z^{2n}\exp(z^2)f(z^2)D_{-2n}(2z)dz$$

and the right hand side is

$$\Gamma(2n)\int_0^\infty v^{n-\frac12}{}_2F_1(n+\tfrac12,\, n;\, \tfrac12;\, -2v)\varphi(1/v)dv.$$

Hence we have

$$\int_0^\infty z^{2n}\exp(z^2)f(z^2)D_{-2n}(2z)dz$$

$$=\frac{(-1)^n}{\pi^{\frac12}2^{n+1}}\int_0^\infty v^{n-\frac12}{}_2F_1(n+\tfrac12,\, n;\, \tfrac12;\, -2v)\varphi(1/v)dv \tag{5}$$

Again let us put $m = n+\frac12$ in (3). Multiplying both sides by $p^{n+3/2}\exp(-\frac14 p^2)D_{2n+1}(p)$ and integrate with respect to p between the limits zero and infinity, we get on using a result due to Mitra (1946, p. 172) and simplifying

$$\int_0^\infty z^{2n+1}\exp(z^2)f(z^2)D_{-2n-3}(2z)dz = \frac{(-1)^n}{\pi^{\frac12}2^{n+1}}\int_0^\infty v^{n+\frac12}{}_2F_1(n+\tfrac32,\, n+2;\, \tfrac32;\, -2v)\varphi(1/v)dv. \tag{5$'$}$$

3.1. We shall now give some simple applications of the above formulae.

In (3) let us put $m = 0$, $p = 2c\cosh t$ and integrate with respect to t between the limits zero and infinity. We get

$$\int_0^\infty zf(z^2)K_0^2(cz)dz = \tfrac14\int_0^\infty \exp(-\tfrac12 c^2v)K_0(\tfrac12 c^2 v)\varphi(1/v)dv \tag{6}$$

Let

$$\varphi(p) = p^{-n}\exp(\tfrac12 c^2/p);\; f(t) = (2t/c^2)^{\frac12 n}I_n(c\sqrt{(2t)})$$

we get

$$\int_0^\infty z^{n+1} I_n(\sqrt{2cz}) K_0^{\,2}(cz)\,dz = \frac{2^{3n/2-2}}{c^{n+2}} \{\Gamma(\tfrac{1}{2}n+\tfrac{1}{2})\}^2, \quad R(n) > -1. \tag{7}$$

3.2. We have the formula (Humbert 1934, p. 12)

$$\int_0^t J_0[2\sqrt{\{x(t-x)\}}]f(x)\,dx \doteqdot \frac{\varphi(p+1/p)}{p+1/p}.$$

Applying this to (4) we get

$$2\int_0^\infty f(t) K_0[2\sqrt{\{t(p+1/p)\}}]\,dt = \int_0^t J_0[2\sqrt{\{x(t-x)\}}]\varphi(1/x)\,dx.$$

Let

$$f(t) = J_0(2\sqrt{(at)}), \quad \varphi(p) = \exp(-a/p).$$

We get

$$2\int_0^\infty J_0(2\sqrt{(at)}) K_0(2\sqrt{\{t(p+1/p)\}})\,dt \doteqdot \int_0^t \exp(-ax) J_0[2\sqrt{\{x(t-x)\}}]\,dx.$$

The left side is $p/(p^2+ap+1)$.

We therefore have

$$\int_0^1 tJ_0[2t\sqrt{\{x(1-x)\}}]\exp(-atx)\,dx \doteqdot p/(p^2+ap+1).$$

Let $a=2$. We know that $p/(p+1)^2 \doteqdot t\exp(-t)$; whence by Lerch's theorem (Lerch, 1903)

$$\int_0^1 \exp(-2tx) J_0[2t\sqrt{\{x(1-x)}]\,dx = \exp(-t) \tag{8}$$

Let

$$f(t) = t^{-\frac{1}{2}}\cos 2\sqrt{(2t)}; \quad \varphi(p) = (\pi p)^{\frac{1}{2}}\exp(-2/p).$$

We obtain

$$2\int_0^\infty t^{-\frac{1}{2}}\cos 2\sqrt{(2t)} K_0[2\sqrt{\{t(p+1/p)\}}]\,dt \doteqdot \pi^{\frac{1}{2}}\int_0^t x^{-\frac{1}{2}}e^{-2x} J_0[2\sqrt{\{x(t-x)\}}]\,dx.$$

The left hand side is

$$\frac{\pi p^{\frac{1}{2}}}{p+1} \doteqdot \tfrac{1}{2}i(D^2_{-1}(i\sqrt{(2t)}) - D^2_{-1}(-i\sqrt{(2t)})) \quad \text{(Mitra, 1933).}$$

Hence again by Lerch's Theorem, we have

$$\int_0^1 x^{-\frac{1}{2}}\exp(-2tx) J_0[2t\sqrt{\{x(1-x)\}}]\,dx = \tfrac{1}{2}i(\pi t)^{-\frac{1}{2}}[D^2_{-1}(i\sqrt{2t})) - D^2_{-1}(-i\sqrt{(2t)})] \tag{8'}$$

More generally, when

$$f(t) = t^{\frac{1}{2}n} J_n(2\sqrt{(2t)}), \quad \varphi(p) = 2^{\frac{1}{2}n}p^{-n}\exp(-2/p),$$

we have

$$\int_0^1 x^n J_0[2t\sqrt{\{x(1-x)\}}]\exp(-2tx)\,dx = \frac{1}{n+1}\,{}_1F_1(2n+2;\ n+2;\ -t) \tag{8''}$$

3.3. It is known that

$$x^{\nu+\frac{1}{2}}\exp\left(\tfrac{1}{4}x^2\right)D_{-2\nu-3}(x) \text{ and } x^{\nu-\frac{1}{2}}\exp\left(\tfrac{1}{4}x^2\right)D_{-2\nu}(x) \text{ are } R_\nu.$$

That is

$$\int_0^\infty x^{n+1}\exp\left(\tfrac{1}{4}x^2\right)J_n(xy)D_{-2n-3}(x)\,dx = y^n\exp\left(\tfrac{1}{4}y^2\right)D_{-2n-3}(y)$$

Now

$$y^{-n}z^{-\frac{1}{2}n}J_n(2y\sqrt{z}) \doteqdot \frac{1}{\Gamma(n+1)}\,{}_1F_1(1;\,n+1;\,-y^2/p).$$

Writing $\frac{1}{2}z$ for z and putting

$$f(z) = y^{-n}z^{-\frac{1}{2}n}J_n(2y\sqrt{x}) \text{ and } \varphi(p) = \frac{1}{\Gamma(n+1)}\,{}_1F_1(1;\,n+1;\,-y^2/p)$$

we get

$$\int_0^\infty v^{n+\frac{1}{2}}\,{}_1F_1(1;\;n+1;\;-y^2v)\,{}_2F_1(n+\tfrac{3}{2},\,n+2;\;\tfrac{3}{2};\;-2v)\,dv$$
$$= (-1)^n\tfrac{1}{2}\pi^{\frac{1}{2}}\Gamma(n+1)\exp\left(\tfrac{1}{4}y^2\right)D_{-2n-3}(y). \tag{9}$$

In a similar manner, with the same values of $f(z)$ and $\varphi(p)$, we get

$$\int_0^\infty v^{n-\frac{1}{2}}\,{}_1F_1(1;\;n+1;\;-y^2v)\,{}_2F_1(n+\tfrac{1}{2},\,n;\;\tfrac{1}{2};\;-2v)\,dv$$
$$= (-1)^n\pi^{\frac{1}{2}}\Gamma(n+1)y^{-1}\exp\left(\tfrac{1}{4}y^2\right)D_{-2n}(y). \tag{10}$$

4. Certain Contour Integrals.

If

$$f(t) = \sum_{n=0}^{\infty}\frac{a_n}{n!}t^n$$

with t real and positive, it follows from an application of the Borel sum formula that

$$\varphi(u) = u\int_0^\infty \exp(-ut)\,f(t)\,dt. \tag{11}$$

If the integral (11) is absolutely convergent when u is real and equal to α, then $\varphi(u)/u$ is an analytic function provided $R(\lambda) > \alpha$ and we have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(\lambda t)\,\frac{\varphi(\lambda)}{\lambda}\,d\lambda, \quad (c > \alpha)$$

since

$$\int_0^\infty \frac{f(t)}{a+t}\,dt = \int_0^\infty \exp(-ax)\,\frac{\varphi(x)}{x}\,dx,$$

we get by an application of this theorem (Mellin, 1902)

$$\frac{\varphi(a)}{a} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(a\lambda)\,d\lambda\int_0^\infty\frac{f(t)\,dt}{\lambda+t}, \quad (c > 0) \tag{12}$$

4.1. Let $f(t) = t^{-\frac{1}{2}}\cos bt^{\frac{1}{2}}$. Then $\varphi(p) = (\pi p)^{\frac{1}{2}}\exp\left(-\tfrac{1}{4}b^2/p\right)$.

Therefore

$$\frac{\varphi(a)}{a} = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\exp(a\lambda)\,d\lambda\int_0^\infty\frac{\cos bt}{\lambda+t^2}\,dt = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty}\exp\left(a\lambda-b\lambda^{\frac{1}{2}}\right)\lambda^{-\frac{1}{2}}\,d\lambda$$

Substituting for $\varphi(a)$, we get

$$\left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left(-\tfrac{1}{4}b^2/a\right) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} \exp\left(a\lambda - b\lambda^{\frac{1}{2}}\right)\lambda^{-\frac{1}{2}}d\lambda$$

Let $a = \frac{1}{2}$. Therefore

$$(2\pi)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}b^2\right) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} \exp\left(\tfrac{1}{2}\lambda - b\lambda^{\frac{1}{2}}\right)\lambda^{-\frac{1}{2}}d\lambda$$

Differentiating n times, we obtain

$$D_n(b) = \frac{1}{2\sqrt{(2\pi)}i} \exp\left(\tfrac{1}{4}b^2\right) \int_{c-i\infty}^{c+i\infty} \exp\left(\tfrac{1}{2}\lambda - b\lambda^{\frac{1}{2}}\right)\lambda^{\frac{1}{2}(n-1)}d\lambda. \tag{13}$$

The path of integration is curved (if necessary) with cuts so that the integrand is a single-valued analytic function.

4.2. Let us multiply both sides of (13) by $\exp\left(-\frac{3}{4}b^2\right)b^m$ and integrate with respect to b between the limits zero and infinity.

Inverting the order of integration,* we get

$$\int_0^\infty \exp\left(-\tfrac{3}{4}b^2\right)b^m D_n(b)db = \frac{1}{2\sqrt{(2\pi)}i}\int_{c-i\infty}^{c+i\infty} \exp\left(\tfrac{1}{2}\lambda\right)\lambda^{\frac{1}{2}(n-1)}d\lambda \int_0^\infty \exp\left(-\tfrac{1}{2}b^2 - b\lambda^{\frac{1}{2}}\right)b^m db$$

$$= \frac{\Gamma(m+1)}{2\sqrt{(2\pi)}i}\int_{c-i\infty}^{c+i\infty} \exp\left(\tfrac{3}{4}\lambda\right)\lambda^{\frac{1}{2}(n-1)}D_{-m-1}\left(\lambda^{\frac{1}{2}}\right)d\lambda.$$

But the left hand side is equal to

$$\frac{\pi^{\frac{1}{2}}2^{\frac{1}{2}n-m-1}\Gamma(m+1)}{\Gamma\left(\frac{1}{2}m-\frac{1}{2}n+1\right)}\,{}_2F_1\left(-\tfrac{1}{2}n,\ \tfrac{1}{2}m+\tfrac{1}{2};\ \tfrac{1}{2}m-\tfrac{1}{2}n+1;\ \tfrac{1}{2}\right).$$

Hence

$$\frac{1}{2\sqrt{(2\pi)}i}\int_{c-i\infty}^{c+i\infty} \exp\left(\tfrac{3}{4}\lambda\right)\lambda^{\frac{1}{2}(n-1)}D_{-m-1}\left(\lambda^{\frac{1}{2}}\right)d\lambda$$

$$= \pi^{\frac{1}{2}}\frac{2^{\frac{1}{2}n-m-1}}{\Gamma\left(\frac{1}{2}m-\frac{1}{2}n+1\right)}\,{}_2F_1\left(-\tfrac{1}{2}n,\ \tfrac{1}{2}m+\tfrac{1}{2};\ \tfrac{1}{2}m-\tfrac{1}{2}n+1;\ \tfrac{1}{2}\right). \tag{14}$$

4.3. Again

$$D_0(b+c)D_n(b+c) = \frac{1}{2\sqrt{(2\pi)}i}\int_{c-i\infty}^{c+i\infty} \exp\left(\tfrac{1}{2}\lambda - b\lambda^{\frac{1}{2}}\right)\sum_{m=0}^{\infty}(-1)^m\frac{\lambda^{\frac{1}{2}(m+n-1)}c^m}{m!}d\lambda$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}D_0(b)D_{m+n}(b)c^m, \tag{15}$$

on term by term integration which can be easily justified.

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* Inversion can be easily justified.

By applying (1) we can easily prove that if

$$\varphi(p) \doteqdot f(t), \text{ then } p/(p+x)^{\frac{1}{2}} \doteqdot \exp(-tx)/\sqrt{(\pi t)} \text{ and so}$$

$$\int_0^\infty \frac{f(x)\,dx}{(t+x)^{\frac{1}{2}}} = \frac{1}{\pi^{\frac{1}{2}}}\int_0^\infty \exp(-tv)v^{-(3/2)}\varphi(v)\,dv.$$

Hence if we write

$$\psi(t) = t\int_0^\infty \frac{f(x)\,dx}{(t+x)^{\frac{1}{2}}};$$

then

$$t^{-3/2}\varphi(t) = \frac{1}{2\pi^{\frac{1}{2}}i}\int_{c-i\infty}^{c+i\infty} \exp(t\lambda)\,d\lambda \int_0^\infty \frac{f(x)\,dx}{(\lambda+x)^{\frac{1}{2}}}. \tag{16}$$

Let

$$f(x) = x^{-\frac{1}{2}}K_0(2ax^{\frac{1}{2}}), \quad \varphi(t) = \tfrac{1}{2}(\pi t)^{\frac{1}{2}}\exp(\tfrac{1}{2}a^2/t)K_0(a^2/2t).$$

Then (16) reduces to

$$\begin{aligned}\varphi(t)t^{-3/2} &= \frac{1}{\pi^{\frac{1}{2}}i}\int_{c-i\infty}^{c+i\infty} \exp(t\lambda)\,d\lambda \int_0^\infty \frac{K_0(2ax)}{(\lambda+x^2)^{\frac{1}{2}}}\,dx \\ &= \frac{\pi^{3/2}}{8i}\int_{c-i\infty}^{c+i\infty} (J_0^2(a\lambda^{\frac{1}{2}})+\gamma_0^2(a\lambda^{\frac{1}{2}}))\exp(t\lambda)\,d\lambda.\end{aligned}$$

Hence we have

$$t^{-1}\exp(\tfrac{1}{2}a^2/t)K_0\left(\frac{a^2}{2t}\right) = \frac{\pi}{4i}\int_{c-i\infty}^{c+i\infty} (J_0^2(a\lambda^{\frac{1}{2}})+\gamma_0^2(a\lambda^{\frac{1}{2}}))\exp(t\lambda)\,d\lambda. \tag{17}$$

We may remark here that the condition for the validity of the changed order of integrations is the same as that for integration under the integral sign when one of the limits is infinite, since the processes are similar. When both the limits are infinite, the conditions are the same as those given in Hobson's Theory of Functions of a Real variable and the Theory of Fourier series, (Hobson, 1907).

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GENERALIZATIONS OF STIELTJES TRANSFORM

By

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1. If we iterate the Laplace transform *i.e.*, if we take

$$f(x) = \int_0^\infty e^{-xt}\varphi(t)dt$$

where

$$\varphi(x) = \int_0^\infty e^{-xt}\psi(t)dt$$

then

$$f(x) = \int_0 \frac{\psi(t)}{x+t}dt \tag{1.1}$$

and the last equation is referred to as Stieltjes transform.

If we replace $\psi(t)dt$ by $d\alpha(t)$ we get the more general case of the equation (1.1) in the form

$$f((t) = \int_0^\infty \frac{d\alpha(t)}{x+t}$$

Dr. R. S. Varma (1952) has given a generalization of Laplace integral in the form

$$f(s) = \int_0^\infty (st)^{m-\frac{1}{2}}e^{-\frac{1}{2}st}W_{k,m}(st)d\alpha(t) \tag{1.2}$$

where $W_{k,m}(z)$ denotes Whittaker function, and in the theory of this generalized Laplace integral he gives two forms for the analogue of Stieltjes transform. In either of these he takes $\alpha(t)$ to be absolutely continuous so that the generalised Laplace integral (1.2) can be written as

$$f(s) = \int_0^\infty (st)^{m-\frac{1}{2}st}W_{k,m}(st)\varphi(t)dt \tag{1.3}$$

If $f(s)$ be the generalized Laplace transform of $\varphi(u)$ in the sense of (1.3) and $\varphi(u)$ be the ordinary Laplace transform of $\psi(t)$ then

$$f(s) = \frac{\Gamma(2m+1)}{s\Gamma(m-k+\frac{3}{2})}\int_0^\infty {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{t}{s}\right]\psi(t)dt \tag{1.4}$$

where $R(2m+1) > 0$. This is the first analogue of Stieltjes transform.

If $f(s)$ be the ordinary Laplace transform of $\varphi(u)$ and $\varphi(u)$ be the generalized Laplace transform of $\psi(t)$, then

$$f(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\int_0^\infty \frac{1}{t}\,{}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{s}{t}\right]\psi(t)dt \tag{1.5}$$

where $R(2m+1) > 0$.

This is the second analogue.

In this paper I have given a few more generalizations of the Stieltjes transform by extending Dr. Varma's results.

2. The first generalization is obtained by iterating the generalized Laplace transform (1.3) and the result can be stated in the form of the following theorem.

Theorem 1. If

$$f(s) = \int_0^\infty (su)^{m-\frac12} e^{-\frac12 su} W_{k,m}(su)\varphi(u)\,du \tag{2.1}$$

and

$$\varphi(u) = \int_0^\infty (ut)^{n-\frac12} e^{-\frac12 ut} W_{l,n}(ut)\psi(t)\,dt \tag{2.2}$$

then

$$\begin{aligned} f(s) = {} & \frac{\Gamma(-2m)\Gamma(2m+1)\Gamma(2m+2n+1)}{\Gamma(\frac12-k-m)\Gamma(2m+n-l+\frac32)} s^{2m} \int_0^\infty t^{-2m-1} \\ & \times {}_2F_1\left[\tfrac12+m+k,\, 2m+2n+1\,;\, 2m+n-l+\tfrac32\,;\, -\frac{s}{t}\right]\psi(t)\,dt \\ & + \frac{\Gamma(2m)\Gamma(2n+1)}{\Gamma(\frac12-k+m)\Gamma(n-l+\frac32)} \int_0^\infty t^{-1}\, {}_3F_2\left[\begin{matrix}\frac12-m+k,\ 1,\ 2n+1 \\ 1-2m,\ n-l+\frac32\end{matrix}\,;\, -\frac{s}{t}\right]\psi(t)\,dt \end{aligned} \tag{2.3}$$

provided that $R(s) > 0$ $R(\rho+n\pm n+1) > 0$, $R(m\pm m+n\pm n+1) > 0$ and $2m \neq$ integer.

and
$$\psi(t) = O(t^\rho) \text{ for small } t,$$
$$= O(e^{-t^\nu}) \text{ for large } t,\ R(\nu) > 0.$$

Proof: Substituting the value of $\varphi(u)$ in (2.1) we have

$$\begin{aligned} f(s) &= \int_0^\infty (su)^{m-\frac12} e^{-\frac12 su} W_{k,m}(su) \int_0^\infty (ut)^{n-\frac12} e^{-\frac12 ut} W_{l,n}(ut)\psi(t)\,dt\,du \\ &= s^{m-\frac12} \int_0^\infty t^{n-\frac12}\psi(t)dt) \int_0^\infty u^{m+n-1} e^{-\frac12 u(s+t)} W_{k,m}(su) W_{l,n}(ut)\,du \end{aligned} \tag{2.4}$$

on changing the order of integration.

Substituting the following expression, in terms of Kummer's function, for $W_{k,m}(su)$,

$$\begin{aligned} W_{k,m}(su) = {} & \frac{\Gamma(-2m)}{\Gamma(\frac12-k-m)} e^{\frac12 su}(su)^{m+\frac12}\, {}_1F_1[\tfrac12+m+k\,;\, 2m+1\,;\, -su] \\ & + \frac{\Gamma(-2m)}{\Gamma(\frac12-k+m)} e^{\frac12 su}(su)^{-m+\frac12}\, {}_1F_1[\tfrac12-m+k\,;\, 1-2m\,;\, -su] \end{aligned}$$

$2m$ not being an integer, we have

$$\begin{aligned} f(s) = s^{m-\frac12} \int_0^\infty t^{n-\frac12}\psi(t)dt \int_0^\infty \sum_{r=0}^\infty \frac{(-1)^r}{r!} & \left[\frac{\Gamma(-2m)}{\Gamma(\frac12-k-m)} \cdot \frac{(\frac12+m+k)_r}{(2m+1)_r} \cdot \frac{s^{m+r+\frac12}}{t^{2m+n+r+\frac12}}\right. \\ & \times (ut)^{2m+n+r-\frac12} e^{-\frac12 ut}\, W_{l,n}(ut) + \frac{\Gamma(2m)}{\Gamma(\frac12-k+m)} \cdot \frac{(\frac12-m+k)_r}{(1-2m)_r} \cdot \frac{s^{-m+r+\frac12}}{t^{n+r+\frac12}} \\ & \left. \times (ut)^{n+r-\frac12} e^{-\frac12 ut}\, W_{l,n}(ut)\right] t\,du \end{aligned}$$

Integrating term by term, which is valid since

(i) the ${}_1F_1$ series is uniformly convergent in the arbitrary interval $(0, A)$ of u,

(ii) $(ut)^{2m+n-\frac{1}{2}}e^{-\frac{1}{2}ut}W_{l,n}(ut)$ and $(ut)^{n-\frac{1}{2}}e^{-\frac{1}{2}ut}W_{l,n}(ut)$ are continuous functions of u for $u \geq 0$ provided that $R(m) \geq 0$, $R(n) \geq 0$,

and (iii) the series obtained upon integration is uniformly convergent in $(0, R)$, with the help of Goldstein's integral (Goldstein, 1932).

$$\int_0^\infty x^{l-1}e^{-(\alpha^2+\frac{1}{2})x}W_{k,m}(x)dx = \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\,{}_2F_1[l+m+\tfrac{1}{2},\ l-m+\tfrac{1}{2};\ l-k+1;\ -\alpha^2]$$
$$R(l \pm m+\tfrac{1}{2}) > 0,\ R(\alpha^2+1) > 0,$$

we get

$$f(s) = s^{m-\frac{1}{2}}\int_0^\infty t^{n-\frac{1}{2}}\psi(t)dt\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left[\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\cdot\frac{(\frac{1}{2}+m+k)_r}{(2m+1)_r}\cdot\frac{s^{m+r+\frac{1}{2}}}{t^{2m+n+r+\frac{1}{2}}}\right.$$
$$\times\frac{\Gamma(2m+r+1)\Gamma(2m+2n+r+1)}{\Gamma(3m+n-l+r+\frac{3}{2})}+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}\cdot\frac{(\frac{1}{2}-m+k)_r}{(1-2m)_r}\cdot\frac{s^{-m+r+\frac{1}{2}}}{t^{n+r+\frac{1}{2}}}$$
$$\left.\times\frac{\Gamma(r+1)\Gamma(2n+r+1)}{\Gamma(n-l+r+\frac{3}{2})}\right],$$

provided that $R(2m+2n+1) > 0$, $R(2m+1) > 0$, $R(2n+1) > 0$.

In order to justify the change in the order of integration in (2.4) let

$$\chi(t) = \psi(t)t^{n-\frac{1}{2}}\int_0^\infty u^{m+n-1}e^{-\frac{1}{2}u(s+t)}W_{k,m}(su)W_{l,n}(ut)du$$

and

$$\beta(u) = u^{m+n-1}W_{k,m}(su)\int_0^A t^{n-\frac{1}{2}}e^{-\frac{1}{2}u(s+t)}W_{l,n}(ut)\psi(t)dt$$

where A is small.

Now $\chi(t)$ is uniformly convergent in $t \geq 0$, if $R(\varrho+n \pm n) \geq 0$ and $R(m \pm m+n \pm n+1) > 0$.

Similarly $\beta(u)$ also converges uniformly in $u \geq 0$ provided that $R(m+n \pm m \pm n) \geq 0$ and $R(n \pm n+\varrho+1) > 0$.

Next let us consider the integral

$$I = \int_T^\infty |\,u^{m+n-1}W_{k,m}(su)\,|\,du\int_{T'}^\infty |\,t^{n-\frac{1}{2}}e^{-\frac{1}{2}u(s+t)}W_{l,n}(ut)\psi(t)\,|\,dt,$$

where T and T' are large

It is easy to see that I does not exceed a constant multiple of

$$\int_T^\infty |\,s^k u^{l+m+n+k-1}e^{-su}\,|\,du\int_{T'}^\infty |\,t^{l+n-\frac{1}{2}}e^{-nt-t^\nu}\,|\,dt$$

which tends to zero provided that $R(s) > 0$, $R(\nu) > 0$.

Hence the change in the order of integration in justified when $R(\nu) > 0$, $R(s) > 0$, $R(n) \geq 0$, $R(m) \geq 0$, $R(\rho) \geq 0$.

By analytic continuation, these conditions can be relaxed to those mentioned in the theorem.

If, however, we substitute the value of $W_{l,n}(ut)$ in terms of Kummer's function in (2.4) and proceed on similar lines, we shall get

$$f(s) = \frac{\Gamma(-2n)\Gamma(2n+1)\Gamma(2m+2n+1)s^{-2n-1}}{\Gamma(\frac{1}{2}-n-l)\Gamma(m+2n-k+\frac{3}{2})}$$

$$\times \int_0^\infty t^{2n}{}_2F_1\left[2n+2m+1,\ n+l+\frac{1}{2}\ ;\ m+2n-k+\frac{3}{2}\ ;\ -\frac{t}{s}\right]\psi(t)dt$$

$$+\frac{\Gamma(2n)\Gamma(2m+1)s^{-1}}{\Gamma(\frac{1}{2}+n-l)\,\Gamma(m-k+\frac{3}{2})}\int_0^\infty {}_3F_2\left[\begin{matrix}2m+1,\ 1,\ l-n+\frac{1}{2}\\ m-k+\frac{3}{2},\ -2n+1\end{matrix}\ ;\ -\frac{t}{s}\right]\psi(t)dt$$

where $R(\nu) > 0$, $R(s) > 0$, $R(\varrho+n\pm n+1) > 0$, $R(m\pm m+n\pm n+1) > 0$.

Corollary 1: If we put $k = \frac{1}{2}-m$ in (2.3), we get

$$f(s) = \frac{\Gamma(2n+1)}{\Gamma(n-l+\frac{3}{2})}\int_0^\infty \frac{1}{t}\,{}_2F_1\left[1,\ 2n+1\ ;\ n-l+\tfrac{3}{2}\ ;\ -\frac{s}{t}\right]\psi(t)\,dt,$$

$$R(2n+1) > 0$$

This is the same as the result (1.5) given by Dr. Varma.

Collorary 2: If we put $l = \frac{1}{2}-n$ in (2.5), we get

$$f(s) = \frac{\Gamma(2m+1)}{s\Gamma(m-k+\frac{3}{2})}\int_0^\infty {}_2F_1\left[2m+1,\ 1\ ;\ m-k+\tfrac{3}{2}\ ;\ -\frac{t}{s}\right]\psi(t)dt,$$

$$R(2m+1) > 0.$$

which is the result (1.4).

3. The second generalization of Stieltjes transform can be obtained by taking $f(s)$ to be the generalized Laplace transform of $\varphi(u)$ as given by (1.2) and $\varphi(u)$ to be the generalization of Laplace transform of $\psi(t)$ in another form given by Dr. Varma (1947). The result can be stated in the following form.

Theorem 2: If

$$f(s) = \int_0^\infty (su)^{m-\frac{1}{2}}e^{-\frac{1}{2}su}W_{k,m}(su)\varphi(u)\,du$$

and

$$\varphi(u) = \int_0^\infty (2tu)^{-\frac{1}{4}}W_{l,n}(2tu)\psi(t)\,dt$$

then

$$f(s) = s^{m-\frac{1}{4}}\int_0^\infty (2t)^{-\frac{1}{4}}\psi(t)dt\left\{\sum_{r=0}^{\infty}\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\cdot\frac{(\frac{1}{2}+m-k)_r}{(2m+1)_r\ r!}\cdot\frac{s^{m+r+\frac{1}{4}}}{(2t)^{2m+r+\frac{3}{4}}}\right.$$

$$\times\frac{\Gamma(2m+n+r+\frac{5}{4})\Gamma(2m-n+r+\frac{5}{4})}{\Gamma(2m-l+r+\frac{7}{4})}{}_2F_1\left[\begin{matrix}2m+n+r+\frac{5}{4},\ 2m-n+r+\frac{5}{4}\\ 2m-l+r+\frac{7}{4}\end{matrix}\ ;\ \frac{1}{2}-\frac{s}{2t}\right]$$

$$+\sum_{r=0}^{\infty}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}\cdot\frac{(\frac{1}{2}-m-k)_r}{(1-2m)_r\ r!}\cdot\frac{s^{-m+r+\frac{1}{4}}}{(2t)^{3/4+r}}\cdot\frac{\Gamma(r+n+\frac{5}{4})\Gamma(r-n+\frac{5}{4})}{\Gamma(r-l+\frac{7}{4})}$$

$$\left.\times{}_2F_1\left[r+n+\tfrac{5}{4},\ r-n+\tfrac{5}{4}\ ;\ r-l+\tfrac{7}{4}\ ;\ \tfrac{1}{2}-\frac{s}{2t}\right]\right\} \quad (3.1)$$

where $R(\varrho\pm n+\frac{5}{4}) > 0$, $R(m\pm m\pm n+\frac{5}{4}) > 0$, $R(s) > 0$, $R(\nu) > 0$, and the behaviour of $\psi(t)$ is given by

$$\begin{aligned}\psi(t) &= O(t^{\rho}) \text{ for small } t\\ &= O(e^{-t^{\nu}}) \text{ for large } t,\ R(\nu)>0.\end{aligned}$$

Proof: Substituting the value of $\varphi(u)$ in $f(s)$, we have

$$\begin{aligned} f(s) &= \int_0^\infty (su)^{m-\frac12} e^{-\frac12 su} W_{k,m}(su) \int_0^\infty (2tu)^{-\frac14} W_{l,n}(2tu)\, \psi(t)\, dt\, du \\ &= s^{m-\frac12} \int_0^\infty (2t)^{-\frac14} \psi(t)\, dt \int_0^\infty u^{m-\frac34} e^{-\frac12 su} W_{k,m}(su)\, W_{l,n}(2ut)\, du \qquad (3.2)\end{aligned}$$

on changing the order of integration,

$$= s^{m-\frac12} \int_0^\infty (2t)^{-\frac14} \psi(t) dt \int_0^\infty e^{-su} \Bigg[s^{m+\frac12} \frac{\Gamma(2m)}{\Gamma(\frac12-k-m)} u^{2m-\frac14}\, {}_1F_1\left(\begin{matrix}\frac12+m-k\\ 2m+1\end{matrix};\ su\right)$$

$$+ s^{-m+\frac12} \frac{\Gamma(2m)}{\Gamma(\frac12-k+m)} u^{-\frac14}\, {}_1F_1(\tfrac12-k-m;\ -2m+1;\ su)\Bigg] W_{l,n}(2ut) du$$

on substituting the expression for $W_{k,m}(su)$ in terms of Kummer's functions.

Writing the equivalent series for ${}_1F_1$ functions and integrating term by term, which is easily justified, we get

$$f(s) = s^{m-\frac12} \int_0^\infty (2t)^{-\frac14} \psi(t) dt \Bigg\{ \sum_{r=0}^{\infty} \frac{\Gamma(-2m)}{\Gamma(\frac12-k-m)} \cdot \frac{(\frac12+m-k)_r}{(2m+1)_r\, r!} \cdot \frac{s^{m+r+\frac12}}{(2t)^{2m+r+\frac34}}$$

$$\times \frac{\Gamma(2m+n+r+\frac54)\Gamma(2m-n+r+\frac54)}{\Gamma(2m-l+r+\frac74)}\, {}_2F_1\left[\begin{matrix}2m+n+r+\frac54,\ 2m-n+r+\frac54\\ 2m-l+r+\frac74\end{matrix};\ \frac12-\frac{s}{2t}\right]$$

$$+ \sum_{r=0}^{\infty} \frac{\Gamma(2m)}{\Gamma(\frac12-k+m)} \cdot \frac{(\frac12-m-k)_r}{(1-2m)_r\, r!} \cdot \frac{s^{-m+r+\frac12}}{(2t)^{\frac34+r}} \cdot \frac{\Gamma(r+n+\frac54)\Gamma(r-n+\frac54)}{\Gamma(r-l+\frac74)}$$

$$\times\, {}_2F_1\left[r+n+\tfrac54,\ r-n+\tfrac54;\ r-l+\tfrac74;\ \frac12-\frac{s}{2t}\right]$$

In order to justify the change in the order of integration in (3.2) let

$$\chi(t) = \psi(t)\, t^{-\frac14} \int_0^\infty u^{m-\frac34} e^{-\frac12 su} W_{k,m}(su) W_{l,n}(2ut) du$$

and

$$\theta(u) = e^{-\frac12 su} u^{m-\frac34} W_{k,m}(su) \int_0^\infty t^{-\frac14} W_{l,n}(2ut)\, \psi(t) dt$$

Now $\chi(t)$ is uniformly convergent in $t \geq 0$, if

$$R(m \pm m \pm n + \tfrac54) > 0,\ R(s) > 0 \text{ and } R(\rho \pm n + \tfrac14) \geq 0.$$

And $\theta(u)$ converges uniformly in $u \geq 0$ provided that

$$R(\rho \pm n + \tfrac54) > 0,\ R(m \pm m \pm n + \tfrac14) \geq 0.$$

Next if we consider the integral

$$I = \int_T^\infty \left| u^{m-\frac{3}{4}} e^{-\frac{1}{2}su} W_{k,m}(su) \right| du \int_{T'}^\infty \left| t^{-\frac{1}{2}} W_{l,n}(2ut)\psi(t) \right| dt$$

where T and T' are large, we find that the integral does not exceed a constant multiple of

$$\int_T^\infty \left| s^k u^{l+m+k-\frac{3}{4}} e^{-su} \right| du \int_{T'}^\infty \left| t^{l-\frac{1}{2}} e^{-ut-t^\nu} \right| dt$$

which tends to zero provided that $R(s) > 0$, $R(\nu) > 0$.

Hence the change in the order of integration is justified under the conditions

$$R(s) > 0,\ R(\nu) > 0,\ R(\varrho \pm n + \tfrac{1}{4}) \geq 0,\ R(m \pm m \pm n + \tfrac{1}{4}) \geq 0,$$

which can be relaxed to those stated in the theorem by the principle of analytic continuation.

Corollary: If we put $\varrho = \frac{1}{2} - m$ in (3.1) we get

$$f(s) = \int_0^\infty \frac{1}{2t} \cdot \frac{\Gamma(\frac{5}{4}+n)\ \Gamma(\frac{5}{4}-n)}{\Gamma(\frac{7}{4}-l)} {}_2F_1\left[\tfrac{5}{4}+n,\ \tfrac{5}{4}-n\,;\ \tfrac{7}{4}-l\,;\ \tfrac{1}{2}-\frac{s}{2t}\right] \psi(t)dt$$

which gives another analogue of Stieltjes transform.

4. The third generalization is obtained by taking $f(s)$ to be the generalized Laplace transform of $\varphi(u)$ in the form given by Meijer and Greenwood (1940) and $\varphi(u)$ to be the generalized Laplace transform of $\psi(t)$. The result is as follows.

Theorem 8: If

$$f(s) = \sqrt{\frac{2s}{\pi}} \int_0^\infty u^{\frac{1}{2}} K_\nu(su)\varphi(u)\,du$$

and

$$\varphi(u) = \int_0^\infty (ut)^{m-\frac{1}{2}} e^{-\frac{1}{2}ut} W_{k,m}(ut)\psi(t)dt$$

then

$$f(s) = \sqrt{s} \int_0^\infty \sum_{r=0}^\infty \left[\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)} \cdot \frac{(\frac{1}{2}+m-k)_r}{(2m+1)_r\ r!} \cdot \frac{\Gamma(2m+r-\nu+\frac{3}{2})\,\Gamma(2m+r+\nu+\frac{3}{2})}{s^{2m+r+3/2}} t^{2m+r} \right.$$
$$\left. \times \frac{P_{\nu-\frac{1}{2}}^{-2m-r-1}(t/s)}{(t^2/s^2-1)^{m+\frac{1}{2}r+\frac{1}{2}}} + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} \cdot \frac{(\frac{1}{2}-m-k)_r}{(-2m+1)_r\ r!} \times \frac{\Gamma(r-\nu+\frac{3}{2})\Gamma(r+\nu+\frac{3}{2})}{s^{r+3/2}} t^r \cdot \frac{P_{\nu-\frac{1}{2}}^{-r-1}(t/s)}{(t^2/s^2-1)^{\frac{1}{2}r+\frac{1}{2}}} \right]$$
$$\times \psi(t)dt \qquad (4.1)$$

provided that $R(\varrho+m\pm m+1) > 0$, $R(m\pm m\pm\nu+\frac{3}{2}) > 0$, $R(\beta) > 0$, $R(s) > 0$,

and
$$\psi(t) = O(t^\rho) \text{ for small } t,$$
$$= O(e^{-t^\beta}) \text{ for large } t,\ R(\beta) > 0.$$

Proof: The proof is similar to that of theorem II.

Corollary: If we put $k = \frac{1}{2} - m$ in (4.1) we have

$$f(s) = \frac{1}{s}\Gamma(\tfrac{3}{2}+\nu)\Gamma(\tfrac{3}{2}-\nu)\int_0^\infty \left(\frac{t^2}{s^2}-1\right)^{-\frac{1}{2}} P_{\nu-\frac{1}{2}}^{-1}\left(\frac{t}{s}\right)\psi(t)dt$$

which reduces to the ordinary Stieltjes transform if we further put $\nu = \frac{1}{2}$.

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TWO THEOREMS ON SELF-RECIPROCAL FUNCTIONS AND A NEW TRANSFORM

By

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1. Gupta (1945) has given two theorems which enable us to derive by processes of differentiation and integration new self-reciprocal functions from the known ones, which are self-reciprocal in the Hankel Transform.

The object of this paper is to generalise these two theorems and to obtain by the help of these, new self-reciprocal functions from the known ones, which are self-reciprocal in the new transforms. In recent papers (Bhatnagar) we have proved in detail that the kernel

$$\varpi_{\mu,\nu}(xs) = (xs)^{\frac{1}{2}}\int_0^\infty J_\nu(t)J_\mu\left(\frac{xs}{t}\right)\frac{dt}{t}$$

plays the role of a transform. We shall call a function $R_{\mu,\nu}$ if it is self-reciprocal in the new transform for which the kernel is $\varpi_{\mu,\nu}(xs)$.

It can be easily shown that

$$\varpi_{n_1,n_2\dots n_n}(xs)$$

defined as

$$\varpi_{n_1,n_2\dots n_n}(xs) = (xs)^{\frac{1}{2}}\int_0^\infty\cdots\int_0^\infty J_{n_1}(t_1)J_{n_2}(t_2)\dots J_{n_{n-1}}(t_{n-1})\times J_{\eta_n}\left(\frac{xs}{t_1\dots t_{n-1}}\right)\frac{dt_1\dots dt_{n-1}}{t_1t_2\dots t_{n-1}}$$

can also be taken as a kernel in the most general transform. In proving our theorems, we shall use this kernel and obtain the generalisations of Gupta's theorems.

2. Hardy and Titchmarsh (1930) have defined a class of functions $A(\omega, a)$, where $0 < \omega \leq \pi$, $a < \frac{1}{2}$, as functions which (i) are analytic functions of $x = re^{i\theta}$, regular in the angle A defined by $r > 0$, $|\theta| < \omega$ and (ii) are $O[|x|^{-a-\delta}]$ for small x and $O[|x|^{a-1+\delta}]$ for large x, for every positive δ and uniformly in any angle $|\theta| \leq \omega-\eta < \omega$. Concerning this class of functions, they have proved a theorem. In the paper referred to above, we have given a similar theorem. Our theorem is

A necessary and sufficient condition that a function $f(x)$ of $A(\omega, a)$ should be $R_{\mu,\nu}$ is that it should be of the form.

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^s\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)x^{-s}\psi(s)ds \tag{1.1}$$

where $\psi(s)$ is regular in

$$a < \sigma < 1-a\,; \tag{1.2}$$

satisfies the identical equation $\psi(s) \equiv \psi(1-s)$ and is

$$O[e^{(\frac{1}{2}\pi-\omega+\eta)|t|}] \tag{1.3}$$

for every positive η *and uniformly in any strip interior to* (1.2) *and* c *is any value of* σ *in* (1.2).

This theorem can be extended when $f(x)$ is $R_{n_1, \ldots n_s}$; the proof being on the same lines.

Theorem I. *Let* E *and* E_1 *stand for the operators*

$$x^{\lambda-1}\int_0^x x^{-\lambda}dx \text{ and } -x^{-\lambda}\int_0^x x^{\lambda-1}dx,$$

respectively, $R(\lambda)$ *lying between* a *and* $1-a$. *Then* $E^mE_1{}^n+E^nE_1{}^m$ *is an operator transforming an* $R_{n_1, n_2 \ldots n_s}$ *into the* $R_{n_1, n_2, \ldots n_s}$ *transform*, where* $E^mE_1{}^n$ *stands for the operation on* E *performed* m *times and the operation on* E_1 *performed* n *times in all, in any order.*

Proof: Let $f(x)$ belong to the class $A(\omega, a)$ and be $R_{n_1, n_2, \ldots n_s}$. Then by the theorem stated in Art. 2.

$$f(x) = \int_{c-i\infty}^{c+i\infty} P(s)x^{-s}ds$$

where

$$P(s) = \frac{1}{2\pi i}2^{\frac{1}{2}ns}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}n_1+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}n_2+\tfrac{1}{2}s)\ldots\Gamma(\tfrac{1}{4}+\tfrac{1}{2}n_n+\tfrac{1}{2}s)\psi(s).$$

and $\psi(s)$ satisfies the condition

$$\psi(s) = \psi(1-s)$$

and

$$\psi(s) = O[e^{(\frac{1}{2}n\pi-\omega+\eta)|t|}].$$

Now

$$Ef(x) = x^{\lambda-1}\int_0^x x^{-\lambda}dx\int_{c-i\infty}^{c+i\infty} P(s)x^{-s}ds$$

$$= x^{\lambda-1}\int_{c-i\infty}^{c+i\infty} P(s)ds\int_0^x x^{-\lambda-s}dx = \int_{c-i\infty}^{c+i\infty}\frac{x^{-s}P(s)ds}{1-\lambda-s}. \tag{3.1}$$

The change in the order in (3.1) is valid for $R(\lambda+s) < 1$ by de la Vallee Poussin's test, on account of the absolute convergence of both the x- and s-integrals; the latter by virtue of the asymptotic behaviour

$$\Gamma(\alpha+i\beta+i\gamma t) = O(e^{-\frac{1}{2}\pi|\gamma|t}t^{\alpha-\frac{1}{2}}) \text{ as } |t|\to\infty;$$

$$|P(s)x^{-s}| = O(e^{(-\omega+\eta)|t|}) \text{ for all } \eta > 0.$$

Repeating the process it is easy to see that in general

$$E^mf(x) = \int_{c-i\infty}^{c+i\infty}\frac{P(s)x^{-s}ds}{(1-s-\lambda)^m}.$$

Similarly it may be shown that provided $R(s-\lambda) < 0$,

$$E_1{}^nf(x) = \int_{c-i\infty}^{c+i\infty}\frac{P(s)x^{-s}ds}{(s-\lambda)^n}.$$

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* That is into another $R_{n_1, n_2, \ldots n_s}$ transform.

It follows therefore that

$$E^m E_1{}^n f(x) = \int_{c-i\infty}^{c+i\infty} \frac{P(s)x^{-s}}{(s-\lambda)^n(1-s-\lambda)^m}\,ds,$$

provided $R(s-\lambda) < 0$ and $R(s+\lambda) < 1$, i.e. $a < R(\lambda) < 1-a$. the order of operations being indifferent Hence

$$(E_1{}^m E^n + E_1{}^n E^m)f(x)$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}ns}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}n_1+\tfrac{1}{2}s)\ldots\Gamma(\tfrac{1}{4}+\tfrac{1}{2}n_n+\tfrac{1}{2}s)\times\psi_1(s)x^{-s}ds \tag{3.2}$$

where

$$\psi_1(s) \equiv \{(1-s-\lambda)^{-m}(s-\lambda)^{-n}+(1-s-\lambda)^{-n}(s-\lambda)^{-m}\}\psi(s)$$

since $\psi_1(s)$ satisfies all the conditions as $\psi(s)$ in Art. 1, the theorem follows on comparing (3.2) with (1.1).

Cor : When $m = n$, we have the theorem :

If $f(x)$ is $R_{n_1, n_2, \ldots n_n}$ then $E^m E_1{}^m f(x)$ is also $R_{n_1, n_2, \ldots n_n}$, provided $a < R(\lambda) < 1-a$.

Theorem II. *Let $\Delta \equiv \lambda + x\dfrac{d}{dx}$ and $\Delta_1 = \lambda - 1 - x\dfrac{d}{dx}$; then $\Delta^m\Delta_1{}^n + \Delta^n\Delta_1{}^m$ is an operator transforming an $R_{n_1, n_2, \ldots n_n}$ into another $R_{n_1, n_2, \ldots n_n}$ transform.*

The proof of the theorem follows the same lines as those of the previous theorem. It may be noted that

$$\Delta\Delta_1 \equiv \left[\lambda + x\frac{d}{dx}\right]\left[\lambda - 1 - x\frac{d}{dx}\right] \equiv \lambda(\lambda-1) - \left(x^2\frac{d}{dx}\right)$$

since the sum of two or more $R_{n_1, n_2, \ldots, n_n}$ transforms is itself a $R_{n_1, \ldots n_n}$ transform, the operator $\Delta\Delta_1 - \lambda(\lambda-1)$ and consequently any polynomial in $(d/dx)(x^2(d/dx))$ will transform an $R_{n_1, n_2, \ldots n_n}$ transform into another $R_{n_1, n_2, \ldots n_n}$ transform.

4. *Example :*

We know that $x^{n+\mu+\frac{1}{2}}K_n(x)$ is $R_{\mu,\nu}$; $\nu = \mu + 2n$, $n \geq 0$.

We can easily prove that

$$\frac{d}{dx}\left[x^2\frac{d}{dx}\left(x^{n+\mu+\frac{1}{2}}K_n(x)\right)\right] = x^{n+\mu+5/2}K_n''(x)$$

$$+ (2n+2\mu+3)x^{n+\mu+3/2}K_n'(x) + (n+\mu+\tfrac{1}{2})(n+\mu+\tfrac{3}{2})x^{n+\mu+\frac{1}{2}}K_n(x).$$

Hence

$$[x^{n+\mu+5/2}K_n''(x) + (2n+2\mu+3)x^{n+\mu+3/2}K_n'(x)] \text{ is } R_{\mu,\nu}.$$

This can also be verified directly.

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FINITE DEFORMATION IN THE INTERIOR OF THE EARTH

By

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Abstract

Application of the finite strain theory has been made to find out the deformation n the interior of the earth considered as a self-gravitating, isotropic, heterogeneous sphere, density within which varies according to Roche's law. Small strain theory was applied by Love to the same problem, density in which was, however, assumed to be uniform. Results of both the theories have been compared.

Introduction

The theory of finite strain has recently received the attention of many workers. It has been applied to various problems which cannot be treated by the classical theory of small deformation, (Seth, 1935, 1950; Murnaghan, 1937, 1951; Truesdell, 1952). It gives good agreement with Bridgman's (1933, 1935) experiments on the compressibility of media under hydrostatic pressure upto 10^5 atmospheres. Its success in solving various problems relating to finite deformation has naturally given rise to the possibility of applying the theory to the discussion of problems relating to the earth.

Applications to seismological problems have been made by Birch (1939) and by Biot (1940). Assuming that the initial stress is hydrostatic and pressure gradient uniform Biot has shown that the influence of the second order effects on seismological waves is negligible, though they may be significant in the case of the earth's tidal movements. Jeffreys (1942) pointed out in discussing elastic instability that the relative importance of rotation and strain need main consideration; departures from linear-strain theory can be significant only when the form of the body is such that there exists possible displacements in which the rotations are large compared with the strains; within the earth such displacements are not possible at places of very great stress.

However. the application of the classical theory to the calculation of strains in the interior of the earth is, as pointed out by Chree (1891), beset with certain difficulties, even if we neglect the effects of rotation and other disturbing forces, as the internal stress is too great to permit the direct application of the superposable small strain theory. One way to avoid this difficulty is to treat the material of the earth as homogeneous and incompressible which, however, does not correspond to physical facts.

The problem of the sphere was developed by Lord Kelvin (1863) who sought to utilise it for the purpose of investigating the rigidity of the earth (Kelvin, 1876) and also by Darwin (1879, 1882) in connection with other problems of cosmic physics. Love

(1944) solved the problem of a homogeneous isotropic sphere held strained by the mutual gravitation of its own parts and showed that the radial strain is contraction within the surface $r = r_0 \{(3-\eta)/(3+3\eta)\}^{\frac{1}{2}}$, but it is extension outside this surface, where r_0 is the radius of the sphere and η Poisson's ratio. That this result cannot be applied to the case of the earth was pointed out by him, the strain inside the earth being too great to admit the application of the classical theory.

In the present paper it is proposed to apply the finite strain theory to find the deformation in the interior of the earth, assuming it to be a self-gravitating isotropic sphere and taking into consideration its heterogeneity of structure, the density varying according to Roche's law (Jeffreys, 1924). As done by Seth (1935) the stress-strain tensor relation is taken as linear.

The body stress equations of eqilibrium

$$\widehat{ij},\, j + F_i = 0$$

refer to the condition in the strained state. In any actual case the boundary conditions are also given in the strained state. Hence, as stressed by Seth (1935) the Eulerian components of strain should be used.

As experimental evidence is lacking, results cannot be verified, but as we can anticipate, we get the dividing zone nearer the centre than what is given by the classical theory, the difference being of the order of 10% and 13% respectively according as we take a particular or a series solution of the differential equation involving β, where $\beta = f(r)$ a function to be defined later on.

Self-gravitating, Heterogeneous, Isotropic sphere.

Assuming the earth to be spherical and at rest only under the mutual gravitation of its own parts, $r = a$ being free from traction, where a is the radius in the strained state, the symmetrical radial displacement can be written as

$$u_r = r(1-\beta) \tag{1}$$

where

$$r^2 = x^2+y^2+z^2 \text{ and } \beta = f(r) \text{ only.}$$

The finite components of strain are given by

$$e_{rr} = \tfrac{1}{2}(1-\beta^2) - \tfrac{1}{2}r^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right) \tag{2.1}$$

$$e_{\theta\theta} = e_{\varphi\varphi} = \tfrac{1}{2}(1-\beta^2) \tag{2.2}$$

$$e_{r\theta} = e_{\theta\varphi} = e_{\varphi r} = 0 \tag{2.3}$$

where $\beta' = d\beta/dr$.

The components of stress are given by

$$\widehat{rr} = \lambda\left[\tfrac{3}{2}(1-\beta^2) - \tfrac{1}{2}r^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right)\right] + \mu\left[(1-\beta^2) - r^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right)\right] \tag{3.1}$$

$$\widehat{\theta\theta} = \widehat{\varphi\varphi} = \lambda\left[\tfrac{3}{2}(1-\beta^2) - \tfrac{1}{2}r^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right)\right] + \mu(1-\beta^2) \tag{3.2}$$

$$\widehat{r\theta} = \widehat{\theta\varphi} = \widehat{\varphi r} = 0.$$

where $\beta' = d\beta/dr$, λ, μ being elastic constants.

The stresses have to satisfy the body-stress equations. The equation

$$\frac{\partial(\widehat{rr})}{\partial r} + \frac{1}{r}\frac{\partial(\widehat{r\theta})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\widehat{r\varphi})}{\partial\varphi} + \frac{1}{r}(2\widehat{rr} - \widehat{\theta\theta} - \widehat{\varphi\varphi} + \widehat{r\theta}\cot\theta) = -\varrho F_r$$

where $\varrho = \sigma(1-kr^2)$ [Roche's law, $\sigma = 10{\cdot}10$, $ka^2 = 0{\cdot}753$], $F_r = -g\frac{r}{a}$

gives
$$\frac{\partial}{\partial r}\left[(3\lambda+6\mu)\beta^2 + (\lambda+2\mu)\left(\beta'^2 + \frac{2\beta\beta'}{r}\right)r^2 + 4\mu\int\beta'^2 r\,dr\right] = -\frac{2g\sigma}{a}(r-kr^3) \tag{4}$$

The other two body-stress equations are identically satisfied.

Hence β which is a function of r alone can be determined from the differential equation

$$(3\lambda+6\mu)\beta^2 + (\lambda+2\mu)r^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right) + 4\mu\int\beta'^2 r\,dr = -\frac{2g\sigma}{a}\left(\frac{r^2}{2} - \frac{kr^4}{4}\right) + k_1$$

where k_1 is a constant of integration.

If we put
$$\eta = \frac{\lambda}{2(\lambda+\mu)} \quad \text{and} \quad c = \frac{1-2\eta}{1-\eta} = \frac{2\mu}{\lambda+2\mu}$$

the above equation becomes

$$3\beta^2 + \left(\beta'^2 + \frac{2\beta\beta'}{r}\right)r^2 + 2c\int\beta'^2 r\,dr = -\frac{g\sigma}{a(\lambda+2\mu)}\left(r^2 - \frac{kr^4}{2}\right) + k_2$$

where k_2 is also a constant.

It may be remarked that since $\frac{1}{2} > \eta > 0$, we have $1 > c > 0$.

Let us again put
$$r^2 = ta^2, \quad P = \frac{g\sigma a}{\lambda+2\mu}$$

and we get
$$2\beta^2 + (2t\beta'+\beta)^2 + 4c\int t\beta'^2 dt = -P(t - \frac{ka^2}{2}t^2) + k_2 \tag{5}$$

where $\beta' = d\beta/dt$.

Determination of β. (a) Particular solution.

The differential equation involving β is a non-linear one. It has not been possible to find an exact solution of the equation. Before proceeding to find a solution in the form of an infinite series, we note that we can get a particular solution in the form

$$\beta = A_0 + A_1 t$$

Substituting this value in (5) we get

$$2(A_0+A_1t)^2 + (A_0+3A_1t)^2 + 4c\int tA_1^2 dt = -P(t - \frac{ka^2}{2}t^2) + k_2$$

Comparing the coeffiecients we get
$$3A_0^2 = k_2 \tag{6.1}$$
$$10A_0A_1 = -P \tag{6.2}$$
$$(11+2c)A_1^2 = P\frac{ka^2}{2} \tag{6.3}$$

Let us now turn to the boundary condition viz. $\widehat{rr} = 0$ at $r = 0$, where a is the radius of the earth in the strained state. Since all the unknown constants in the solution are definitely determined, the satisfaction of the boudary condition will give us a relation between λ and μ. By (3 1)

$$\widehat{rr} = \lambda\left[\tfrac{3}{2}(1-\beta^2)-\tfrac{1}{2}r^2\left(\beta'^2+\frac{2\beta\beta'}{r}\right)\right]+\mu\left[1-\beta^2-r^2\left(\beta'^2+\frac{2\beta\beta'}{r}\right)\right] = 0 \text{ at } r = a$$

or

$$A_1{}^2(11-2c)+A_0A_1(10-4c)+(3-2c)(A_0{}^2-1) = 0 \text{ at } r = a$$

or

$$\frac{g\sigma a}{\lambda+2\mu}.\frac{11-2c}{11+2c}.\frac{ka^2}{2}-\frac{g\sigma a}{\lambda+2\mu}.\frac{10-4c}{10}+(3-2c)\left\{\frac{2g\sigma(11+2c)}{ka'\lambda+2\mu)10^2}-1\right\} = 0. \tag{7}$$

Taking $\eta=0.5(\mu=0)$, $c=0$, $g=981.262$ cm./sec^2. (value at Potsdam) $a=6371$km., $\sigma = 10.10$, $ka^2 = 0.753$ (mean density of the earth $= 5.526$), we get

$$\lambda = 0.53\times10^{12} \text{ dynes/cm.}^2 \tag{8.1}$$

Taking $\eta = 0.33$, $c = 0.507$, we get

$$\lambda+2\mu = 0.49\times10^{12} \text{ dynes/cm.}^2 \tag{8.2}$$

According to seismological observations within the earth the value of λ varies from 0.74×10^{12} dynes/cm.^{2} to 12.6×10^{12} dynes/cm.^{2} from the surface upto a depth of 2898km., below which is assumed the existence of fluid layer, mainly from the fact that S-waves are not propagated through it; μ varies from 0.63×10^{12} dynes/cm.^{2} to 3.03×10^{12} dynes/cm.^{2} and η from 0.269 to 0.5. (Bullen, 1947).

At the surface the theoretical value obtained here is nearly one-fourth of the value obtained by seismological observations.

Case I. $\eta = 0{\cdot}5(\mu = 0)$, $c = 0$

By (6)

$$\beta = 1{\cdot}867-0{\cdot}638\frac{r^2}{a^2} \tag{9.1}$$

By (2.1) the radial strain is given by

$$\begin{aligned} e_{r_1} &= \tfrac{1}{2}\{1-(\beta+r\beta')^2\} \\ &= \frac{1}{2}\left\{1-\left(A_0+3A_1\frac{r^2}{a^2}\right)^2\right\} \\ &= \frac{1}{2}\left\{1-\left(1{\cdot}867-1{\cdot}914\frac{r^2}{a^2}\right)^2\right\}; \end{aligned}$$

radial strain is contraction within the surface $r = 0{\cdot}673a$ and extension outside this surface.

Case II. $\eta = 0{\cdot}33$; $c = 0{\cdot}507$

$$\beta = 2{\cdot}029-0{\cdot}625\frac{r^2}{a^2}\cdot \tag{9·2}$$

The dividing zone is given by $r = 0{\cdot}785a$, In Case I the classical theory gives the dividing zone as $r = 0{\cdot}745a$ and in Case II as $r = 0{\cdot}818a$.

The particular solution of the problem by the application of the finite strain theory, therefore, gives the dividing zone nearer the centre, the order of difference being 10%.

(b) Solution in Series.

Assume $\beta = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \dots + A_n t^n + \dots$

Substituting this value in

$$2\beta^2 + (2t\beta' + \beta)^2 + 4c \int t\beta'^2 dt = -P\left(t - \frac{ka^2}{2} t^2\right) + k_2 \tag{5}$$

where $\beta' = d\beta/dt$, we get

$$\begin{aligned} &2(A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4 + \dots)^2 \\ &\quad + (A_0 + 3A_1 t + 5A_2 t^2 + 7A_3 t^3 + 9A_4 t^4 + \dots)^2 \\ &\quad + 4c \int t(A_1 + 2A_2 t + 3A_3 t^2 + 4A_4 t^3 + \dots)^2 dt = -P\left(t - \frac{ka^2}{2} t^2\right) + k_2 \end{aligned}$$

Comparing the coeffecients we get

$$\left.\begin{aligned} 3A_0^2 &= k_2. \\ 10A_0 A_1 &= -P \\ 14A_0 A_2 + (11+2c)A_1^2 &= \frac{Pka^2}{2} \\ 18A_0 A_3 + \left(34 + 16\frac{c}{3}\right) A_1 A_2 &= 0 \\ (4c+27)A_2^2 + 22A_0 A_4 + (46+6c)A_1 A_3 &= 0 \end{aligned}\right\} \tag{10}$$

and so on.

We express all the coeffecients in terms of A_0. By (10)

$$\left.\begin{aligned} \frac{A_1}{A_0} &= -\frac{P}{10} \cdot \frac{1}{A_0^2} \\ \frac{A_2}{A_0} &= \frac{1}{14}\left\{\frac{Pka^2}{2} \frac{1}{A_0^2} - (11+2c)\frac{P^2}{10^2} \frac{1}{A_0^4}\right\} \\ \frac{A_3}{A_0} &= \frac{1}{18.10.14}\left(34 + 16\frac{c}{3}\right)\left\{\frac{P^2 ka^2}{2} \frac{A_0^4}{A_0^4} - (11+2c)\frac{P^3}{10^2} \frac{1}{A_0^6}\right\} \\ \frac{A_4}{A_0} &= \frac{1}{22}\left[(46+6c)\frac{1}{18.10^2.14}\left(34 + 16\frac{c}{3}\right)\left\{\frac{P^3 ka^2}{2} \frac{1}{A_0^6} - (11+2c)\frac{P^4}{10^2} \frac{1}{A_0^8}\right\}\right. \\ &\quad \left. + (4c+27)\frac{1}{14^2}\left\{\frac{P^2 k^2 a^4}{4} \cdot \frac{1}{A_0^4} - P^3 ka^2 \frac{(11+2c)}{10^2} \cdot \frac{1}{A_0^6} + \frac{(11+2c)^2}{10^4} \frac{P^4}{A_0^8}\right\}\right] \end{aligned}\right\} \tag{11}$$

All the constants except one are then determined. The unknown constant is to be obtained from the boundary condition. For that we have to take some tentative values of the elastic constants. The variation of these constants within the earth have already been mentioned. However, we make an assumptien about the mean value of $\lambda+2\mu (= 9 \times 10^{12}$ dynes/cm.^{2}) from the known values of λ and μ obtained by seismological observations.

Substituting then numerical values in (11) we get (corresponding to Case I, $\eta = 0{\cdot}5 (\mu = 0)$, $c = 0$) retaining figures correct to four decimal places, as the mean density of the earth is known correctly upto three significant figures,

$$\left.\begin{aligned} \frac{A_1}{A_0} &= -{\cdot}0702\,\frac{1}{A_0^{\,2}} \\ \frac{A_2}{A_0} &= {\cdot}0189\,\frac{1}{A_0^{\,2}} - {\cdot}0039\,\frac{1}{A_0^{\,4}} \\ \frac{A_3}{A_0} &= {\cdot}0025\,\frac{1}{A_0^{\,4}} - {\cdot}0005\,\frac{1}{A_0^{\,6}} \\ \frac{A_4}{A_0} &= -{\cdot}0004\,\frac{1}{A_0^{\,4}} + \frac{{\cdot}0006}{A_0^{\,6}} - \frac{{\cdot}0001}{A_0^{\,8}} \end{aligned}\right\} \tag{11a}$$

and so on.

Therefore
$$\beta = A_0\Big(1+\frac{A_1}{A_0}\,t+\frac{A_2}{A_0}\,t^2+\frac{A_3}{A_0}\,t^3+\frac{A_4}{A_0}\,t^4+\ldots\Big)$$
$$= A_0\Big(1-0513\,\frac{1}{A_0^{\,2}}-{\cdot}0018\,\frac{1}{A_0^{\,4}}+\frac{{\cdot}0001}{A_0^{\,6}}+\ldots\Big)$$

when $t = 1$. The numerical value in the last term is doubtful as other terms containing $1/A_0^{\,6}$ have not been taken into account.

By (3·1) the boundary conditian can be written in the form
$$3-2c = (2t\beta'+\beta)^2+2(1-c)\beta^2]_{t=1}.$$

Substituting numerical values and retaining terms upto $1/A_0^{\,4}$
$$\frac{1}{A_0^{\,2}} = 1-\frac{0{\cdot}146}{A_0^{\,2}}-\frac{0{\cdot}0003}{A_0^{\,4}}+\ldots$$

By the method of iteration, taking positive value of square root
$$A_0 = 1{\cdot}070,\ \ A_1 = -0{\cdot}066,\ \ A_2 = 0{\cdot}015,\ \ A_3 = 0{\cdot}002$$
$$\beta = \Big(1{\cdot}070-0{\cdot}066\,\frac{r^2}{a^2}+0{\cdot}015\,\frac{r^4}{a^4}+\ldots\Big) \tag{12.1}$$

neglecting higher order terms.

The dividing zone is given by $r = 0{\cdot}648a$. The negative value of square root leads to the same result.

Proceeding in the same way we get corresponding to Case II

$$A_0 = 1{\cdot}080, \quad A_1 = -0{\cdot}065, \quad A_2 = 0{\cdot}014, \quad A_3 = 0{\cdot}002$$

$$\beta = \left(1{\cdot}080 - 0{\cdot}065\frac{r^2}{a^2} + 0{\cdot}014\frac{r^4}{a^4} + \cdots\right). \tag{12.2}$$

The dividing zone is given by $r = 0{\cdot}707a$

As $r^2/a^2 < 1$ and the coefficients $A_0, A_1, \ldots$ rapidly diminish and also from physical considerations we may expect the β series to be convergent.

The series solution gives the dividing zone 4% nearer the centre compared to the particular solution and 13% nearer the centre compared to the solution obtained from the classical theory.

The depth from the surface of the earth below which the fluid layer is assumed to be situated is 2898km. and the radius of the earth consdired as a sphere is 6371km. Therefore, the fluid layer extends from the centre upto a distance $r = 0{\cdot}545a$. The dividing zone in any case lies beyond this distance from the centre.

In conclusion, I wish to thank Professor B. R. Seth for his helpful guidance during the course of the work.

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ON AN EXTENSION OF BLASIUS THEOREM

By

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(*Communicated by Dr. S. Ghosh—Received July 22, 1952*)

It is known that in the case of two-dimensional steady irrotational motion of an infinite liquid round a solid cylinder, the resultant thrust of the liquid on the cylinder can be calculated easily with the help of Blasius Theorem (Milne-Thomson, p. 163). In this note it is proposed to show that in the case of rotational motion also, it is possible to calculate the resultant thrust by an extension of Blasius Theorem.

Let X, Y be the components of the resultant thrust along the axes and M the moment about the origin. If l, m be the direction cosines of the outward drawn normal to the boundary of the cylinder and p be the pressure then

$$X = -\int pl ds = -\int pdy, \quad Y = -\int pm ds = \int pdx \tag{1}$$

$$X - iY = -i\int p d\bar{z} \tag{2}$$

and

$$M = \int p(mx - ly)ds = \int p(xdx + ydy) \tag{3}$$

the integrals being all taken round the contour of the cylinder.

In steady rotational motion, the pressuse at any point is given by

$$p = c - \tfrac{1}{2}\rho q^2$$

where c is a constant along a stream line. Since the boundary of the cross section is a stream-line, c is constant on the boundary. Substituting in (2) and (3), we get easily

$$X - iY = \tfrac{1}{2}i\rho\int_c q^2 d\bar{z} \tag{4}$$

$$M = -\tfrac{1}{2}\rho\int_C q^2(xdx + ydy) = -\tfrac{1}{2}\rho\mathcal{R}\int_C q^2 z d\bar{z} \tag{5}$$

where $\mathcal{R}$ denotes the real part.

If ψ be the stream function of the liquid motion,

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}.$$

Therefore

$$(u + iv)d\bar{z} = -\frac{\partial\psi}{\partial y}dx + \frac{\partial\psi}{\partial x}dy + id\psi$$

$$(u-iv)dz = -\frac{\partial\psi}{\partial y}dx + \frac{\partial\psi}{\partial x}dy - id\psi$$

On the boundary C of the cylinder $d\psi = 0$, so that on C

$$(u+iv)d\bar{z} = (u-iv)dz$$

and therefore

$$q^2 d\bar{z} = (u+iv)(u-iv)d\bar{z} = (u-iv)^2 dz.$$

Hence

$$X - iY = \tfrac{1}{2} i\varrho \int_C (u-iv)^2 dz \tag{6}$$

$$M = -\tfrac{1}{2}\varrho \mathcal{R} \int_C (u-iv)^2 z\,dz. \tag{7}$$

The formulae (6) and (7) are extensions to the case of rotational motion of the corresponding formulae given by Blasius for irrotational motion These formulae have not the same advantage as those of Blasius, for the integrands in (6) and (7) are not analytic functions of z. But this difficulty can be overcome by replacing the integrands by analytic functions which reduce to the integrands on the boundary C.

Let the boundary of the cylinder be given by the curve $\xi = \xi_0$ in the transformation

$$z = f(\zeta), \quad \zeta = \xi + i\eta$$

When the liquid motion outside the cylinder is known, $u-iv$ is known as a function of z and $\bar{z}$, and therefore as a function of ζ and $\bar{\zeta}$ Now, on the boundary of the cylinder $\bar{\zeta} = 2\xi_0 - \zeta$, so that on the boundary $u-iv$ can be replaced by an analytic function of ζ and therefore of z.

As a simple application let us consider the motion of a liquid of uniform vorticity round an elliptic cylinder. In this case, we use the transformation

$$z = c\cosh\zeta$$

and let $\xi = \xi_0$ be the boundary of the ellipse If the velocity at infinity is due to shear motion parallel to x-axis, given by

$$u = -U - 2\omega y, \quad v = 0$$

and if we have in addition, a circulation κ round the cylinder, then

$$u - iv = -U + i\omega z - i\omega c\cosh\bar{\zeta} - \left[Ae^{-\zeta} - 2iBe^{-2\zeta} + \frac{i\kappa}{2\pi}\right]\frac{1}{c\sinh\zeta}$$

where

$$A = -Uce^{\xi_0}\sinh\xi_0 = -U\,\frac{b}{c}(a+b)$$

$$B = -\tfrac{1}{4}\omega c^2 e^{2\xi_0}(1-\cosh 2\xi_0)$$

$$a = c\cosh\xi_0, \quad b = c\sinh\xi_0.$$

This complex velocity satisfies the equation of continuity, gives 2ω as the vorticity at any point of the liquid, satisfies the condition at infinity and makes the boundary of the ellipse a stream line.

On the boundary $\xi = \xi_0$, $\bar{\zeta} = 2\xi_0 - \zeta$, so that on the boundary

$$u - iv = -U + i\omega z - i\omega c \cosh(2\xi_0 - \zeta) - \left[Ae^{-\zeta} - 2iBe^{-2\zeta} + \frac{i\kappa}{2\pi}\right]\frac{1}{c\sinh\zeta}.$$

Now,

$$e^{-\zeta} = \cosh\zeta - \sinh\zeta = \frac{1}{c}(z - (z^2 - c^2)^{\frac{1}{2}}), \quad e^{-2\zeta} = \frac{1}{c^2}(z - (z^2 - c^2)^{\frac{1}{2}})^2.$$

Hence, on the boundary

$$u - iv = -U + i\omega z - i\omega \cosh 2\xi_0 . z + i\omega \sinh 2\xi_0 (z^2 - c^2)^{\frac{1}{2}}$$

$$-\left[\frac{A}{c}\left(\frac{z}{(z^2 - c^2)^{\frac{1}{2}}} - 1\right) - 2i\frac{B}{c^2}\left(\frac{z^2}{(z^2 - c^2)^{\frac{1}{2}}} - 2z + (z^2 - c^2)^{\frac{1}{2}}\right) + \frac{i\kappa}{2\pi}\frac{1}{(z^2 - c^2)^{\frac{1}{2}}}\right] = f(z).$$

Substituting in (6) and (7), we observe that the integrals can be taken over a large circle Σ with centre as origin, as there is no singularity of the integrands between C and Σ. We have

$$\frac{z}{(z^2 - c^2)^{\frac{1}{2}}} = 1 + \frac{c^2}{2z^2} + \frac{3}{8}\frac{c^4}{z^4} + \dots, \quad (z^2 - c^2)^{\frac{1}{2}} = z - \frac{1}{2}\frac{c^2}{z} - \frac{1}{8}\frac{c^4}{z^3} + \dots,$$

Expanding in powers of $1/z$,

$$u - iv = -U + i\omega z - i\omega z \cosh 2\xi_0 + i\omega \sinh 2\xi_0\left(z - \frac{1}{2}\frac{c^2}{z} - \frac{1}{8}\frac{c^4}{z^3}\dots\right)$$

$$-\left[\frac{A}{c}\cdot\frac{c^2}{2z^2} - 2i\frac{B}{c^2}\cdot\frac{1}{4}\frac{c^4}{z^3} + \frac{i\kappa}{2\pi}\left(\frac{1}{z} + \frac{c^2}{2z^3}\right)\right]\dots$$

$$= i\omega(1 - \cosh 2\xi_0 + \sinh 2\xi_0)z - U - \frac{i}{2z}\left(\omega c^2 \sinh 2\xi_0 + \frac{\kappa}{\pi}\right) - \frac{Ac}{2z^2} + \dots$$

$$= i\omega\frac{2b}{c^2}(a - b)z - U - \frac{i}{2z}\left(\omega 2ab + \frac{\kappa}{\pi}\right) - \frac{Ac}{2z^2} + \dots \tag{8}$$

$$(u - iv)^2 = \dots - 2Aci\omega\frac{b(a - b)}{c^2 z} + i\frac{U}{z}\left(2\omega ab + \frac{\kappa}{\pi}\right) + \frac{1}{z^2} \times \text{real terms}.$$

Now

$$X - iY = \tfrac{1}{2}i\rho \times 2\pi i\left[iU\left(2\omega ab + \frac{k}{\pi}\right) - i\frac{\omega}{c}2Ab(a - b)\right]$$

$$= -i\pi\rho\left[U\left(2\omega ab + \frac{k}{\pi}\right) + 2\omega b^2 U\right] = -i\pi\rho\left[\frac{kU}{\pi} + U2\omega b(a + b)\right]$$

Hence

$$X = 0, \quad Y = \rho U k + \pi\rho\omega U 2b(a + b).$$

We now observe that circulation immediately round the cylinder is given by

$$\int_C (u\,dx + v\,dy) = \int_C\left(-\frac{\partial\psi}{\partial y}dx + \frac{\partial\psi}{\partial x}dx\right) = \int_C (u - iv)dz.$$

From (8) it is seen that the integrand has a pole at the origin with residue

$$-\frac{i}{2}\left(2\omega ab+\frac{k}{\pi}\right)$$

∴ circulation immediately round the cylinder

$$= 2\pi i\left\{-\frac{i}{2}\left(2\omega ab+\frac{\kappa}{\pi}\right)\right. = 2\pi\omega ab+\kappa = \kappa' \text{ (say)}$$

Then

$$Y = \varrho U\kappa' + 2\pi\varrho\omega U b^2$$

and lastly

$$M = -\tfrac{1}{2}\varrho\mathcal{R}\int_C (u-iv)^2 z\,dz = -\tfrac{1}{2}\varrho\mathcal{R}\,2\pi i \times \text{real terms} = 0.$$

Thus we see that the force exerted is a lift force and the moment about the centre of the cylinder is zero.

In conclusion, I wish to express my thanks to Dr. S. Ghosh D Sc , for his continuous interest and kind advice regarding the presentation of this communication.

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ON THE GENERALISED HYPERGEOMETRIC FUNCTION OF APPELL'S TYPE

By

Shanti Saran, *Lucknow, India*

(*Communicated by Dr. S. C Mitra—Received May 6, 1952*)

The object of the present note is to obtain certain generalisations of Hypergeometric Functions of Appell's type (Appell et Kampé de Fériet, 1926) by increasing the number of parameters (Burchnall and Chaundy, 1940; Bailey, 1929, 1934), the number of variables being three. We have obtained relations between them and Appell's and ordinary Hypergeometric Functions and have also obtained partial differential equations satisfied by them.

1. Let us write

$$\varphi_1(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_s(\alpha_3)_t(\beta_1)_r(\beta_2)_s(\beta_3)_t}{(\gamma_1)_r(\gamma_2)_s(\gamma_3)_t\, r!\, s!\, t!} x^r y^s z^t \qquad (1')$$

$$\varphi_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3; \gamma_1; x, y, z)$$

$$= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_s(\alpha_3)_t(\beta_1)_r(\beta_2)_s(\beta_3)_t}{(\gamma_1)_{r+s+t}\, r!\, s!\, t!} x^r y^s z^t \qquad (2')$$

$$\varphi_3(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2; x, y, z)$$

$$= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_s(\alpha_3)_t(\beta_1)_r(\beta_2)_s(\beta_3)_t}{(\gamma_1)_r(\gamma_2)_{s+t}\, r!\, s!\, t!} x^r y^s z^t \qquad (3')$$

$$\varphi_4(\alpha_1, \alpha_2, \alpha_3; \beta_1; \gamma_1; x, y, z)$$

$$= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_s(\alpha_3)_t(\beta_1)_{r+s+t}}{(\gamma_1)_{r+s+t}\, r!\, s!\, t!} x^r y^s z^t \qquad (4')$$

$$\varphi_5(\alpha_1, \alpha_2, \alpha_3; \beta_1; \gamma_1, \gamma_2; x, y, z)$$

$$= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_s(\alpha_3)_t(\beta_1)_{r+s+t}}{(\gamma_1)_r(\gamma_2)_{s+t}\, r!\, s!\, t!} x^r y^s z^t \qquad (5')$$

$$\varphi_6(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_s(\alpha_3)_t(\beta_1)_r(\beta_2)_{s+t}}{(\gamma_1)_r(\gamma_2)_s(\gamma_3)_t\, r!\, s!\, t!} x^r y^s z^t \qquad (6')$$

$$\varphi_7(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z)$$

$$= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_s(\alpha_3)_t(\beta_1)_r(\beta_2)_{s+t}}{(\gamma_1)_r(\gamma_2)_{s+t}\, r!\, s!\, t!} x^r y^s z^t \qquad (7')$$

$$\varphi_8(\alpha_1, \alpha_2;\ \beta_1, \beta_2;\ \gamma_1, \gamma_2;\ x, y, z)$$

$$= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_{s+t}(\beta_1)_r(\beta_2)_{s+t}}{(\gamma_1)_r(\gamma_2)_{s+t}\, r!\, s!\, t!} x^r y^s z^t \tag{8'}$$

$$\varphi_9(\alpha_1;\ \beta_1;\ \gamma_1;\ x, y, z)$$

$$= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_1)_{r+s+t}(\beta_1)_{r+s+t}}{(\gamma_1)_{r+s+t}\, r!\, s!\, t!} x^r y^s z^t \tag{9'}$$

The series (1), (2), (3), (4), (7) and (9) are absolutely convergent if $|x| < 1$, $|y| < 1$ and $|z| < 1$. The series (5) is absolutely convergent if $|x|+|y| < 1$ and $|z| < |y|$ or $|x|+|z| < 1$ and $|y| < 1$. The series (6) and (8) are absolutely convergent if $|x| < 1$ and $|y|+|z| < 1$.

We use the symbolic operators (Burchnall and Chaundy, 1940)

$$\nabla(h) = \frac{\Gamma(h)\Gamma(\delta+\delta'+h)}{\Gamma(\delta+h)\Gamma(\delta'+h)},\quad \Delta(h) = \frac{\Gamma(\delta+h)\Gamma(\delta'+h)}{\Gamma(h)\Gamma(\delta+\delta'+h)}$$

We shall simply state the results without proof. The results can be verified easily.

$$\varphi_2(\alpha_1, \alpha_2, \alpha_2;\ \beta_1, \beta_2, \beta_3;\ \gamma_1;\ x. y, z)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-)^s(\alpha_1)_r(\alpha_2)_s(\beta_1)_r(\beta_2)_s(\beta_3)_s}{(\gamma_1)_{r+2s}\, r!\, s!}$$

$$\times F^{(1)}(\alpha_2+s;\ \beta_2+s, \beta_3+s;\ \gamma_1+r+2s;\ y_1 z)^* \tag{1}$$

$$\varphi_3(\alpha_1, \alpha_1, \alpha_2;\ \beta_1, \beta_2, \beta_3;\ \gamma_1, \gamma_2;\ x, y, z)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-)^s(\alpha_1)_s(\alpha_2)_r(\beta_1)_s(\beta_2)_r(\beta_3)_s}{(\gamma_1)_s(\gamma_2)_{r+s}\, r!\, s!} x^s y^r z^s$$

$$\times F^{(1)}[\alpha_1+s;\ \beta_1+s, \beta_3+s;\ \gamma_1+s, \gamma_2+r+s;\ x, z] \tag{2}$$

$$\varphi_3(\alpha_1, \alpha_1, \alpha_2;\ \beta_1, \beta_1, \beta_2;\ \gamma_1, \gamma_2;\ x, y, z)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_s(\beta_1)_r(\beta_2)_s(-\alpha_1-\beta_1-1)_r}{(\gamma_1)_r(\gamma_2)_{r+s}\, r!\, s!} x^r y^s z^r$$

$$\times F^{(4)}[\alpha_1+r;\ \beta_1+r;\ \gamma_1+r, \gamma_2+s+r;\ x-xz, z-xz] \tag{3}$$

$$\varphi_4(\alpha_1, \alpha_2, \alpha_3;\ \beta_1;\ \gamma_1;\ x, y, z)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_s(\alpha_3)_s(\beta_1)_{r+s}}{(\gamma_1)_{r+2s}\, r!\, s!} x^r y^s z^s$$

$$\times F^{(3)}[\alpha_2+s, \alpha_3+s;\ \beta_1+r+s, \beta_1+r+s;\ \gamma_1+r+2s;\ y, z] \tag{4}$$

$$\varphi_5(\alpha_1, \alpha_1, \alpha_2;\ \beta_1;\ \gamma_1, \gamma_2;\ x, y, z)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-)^r(\alpha_1)_r(\alpha_2)_s(\beta_1)_{s+2r}}{(\gamma_1)_r(\gamma_2)_{r+s}\, r!\, s!} x^r y^s z^r$$

$$\times F^{(4)}[\beta_1+s+2r;\ \alpha_1+r;\ \gamma_1+r, \gamma_2+r+s;\ x, z] \tag{5}$$

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* F's are Appell's Functions.

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_s(\beta_1)_{s+r}(-s-\beta_1-1)_r}{(\gamma_1)_r(\gamma_2)_{s+r}\, r!\, s!}x^s y^r z^r$$

$$\times F^{(4)}[\beta_1+r+s;\ \alpha_1+r;\ \gamma_1+r,\ \gamma_2+r+s;\ y-yz,\ z-yz] \quad (6)$$

$\varphi_4(\alpha_1, \alpha_2, \alpha_3;\ \beta_1;\ \gamma_1;\ x, y, z)$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^s(\alpha_1)_r(\alpha_2)_s(\alpha_3)_s(\beta_1)_{r+2s}x^r y^s z^s}{(\gamma_1+r+s-1)_s(\gamma_1)_{r+2s}\, r!\, s!}$$

$$\times F^{(2)}[\beta_1+r+2s;\ \alpha_2+s,\ \alpha_3+s,\ \gamma_1+r+2s,\ \gamma_1+r+2s,\ y, z] \quad (7)$$

$\varphi_6(\alpha_1, \alpha_2, \alpha_2;\ \beta_1, \beta_2;\ \gamma_1, \gamma_2, \gamma_3;\ x, y, z)$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^r(\alpha_2)_r(\alpha_1)_s(\beta_1)_r(\beta_2)_{r+s}}{(\gamma_1)_r(\gamma_2)_s(\gamma_3)_r\, r!\, s!}x^r y^s z^r$$

$$\times F^{(2)}[\alpha_2+r;\ \beta_1+r,\ \beta_2+r+s;\ \gamma_1+r,\ \gamma_3+r,\ x, z] \quad (8)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_2)_r(\alpha_1)_s(\beta_1)_s(\beta_2)_r(-\beta_1-1)_r}{(\gamma_1)_s(\gamma_2)_r(\gamma_3)_r\, r!\, s!}x^s y^r z^r$$

$$\times F^{(4)}[\beta_2+r;\ \alpha_2+r;\ \gamma_2+r,\ \gamma_3+r;\ y-yz,\ z-yz] \quad (9)$$

$\varphi_6(\alpha_1, \alpha_2, \alpha_3;\ \beta_1, \beta_2;\ \gamma_1, \gamma_1, \gamma_2,\ x, y. z)$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_2)_s(\alpha_1)_r(\alpha_3)_r(\beta_1)_r(\beta_2)_{r+s}}{(\gamma_3)_s(\gamma_1)_r(\gamma_1)_{2r}\, r!\, s!}x^r y^s z^r$$

$$\times F^{(3)}[\alpha_1+r,\ \alpha_3+r;\ \beta_1+r,\ \beta_2+r+s,\ \gamma_1+2r;\ x, z] \quad (10)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_s(\alpha_2)_r(\alpha_3)_r(\beta_1)_s(\beta_2)_{2r}}{(\gamma_1)_r(\gamma_1)_{2r}(\gamma_2)_s\, r!\, s!}x^s y^r z^r$$

$$\times F^{(1)}[\beta_2+2r;\ \alpha_2+r,\ \alpha_3+r;\ \gamma_1+2r;\ y, z] \quad (11)$$

$\varphi_6(\alpha_1, \alpha_1, \alpha_2;\ \beta_1, \beta_2;\ \gamma_1, \gamma_1. \gamma_2;\ x, y, z)$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_2)_s(\alpha_1)_r(\beta_1)_r(\beta_2)_{r+s}(\alpha_1-\gamma_1)_r}{(\gamma_1)_r(\gamma_1)_{2r}(\gamma_3)_s\, r!\, s!}x^r y^s z^r$$

$$\times F^{(1)}[\alpha_1+r;\ \beta_1+r,\ \beta_2+r+s;\ \gamma_1+2r;\ x, z] \quad (12)$$

$\varphi_6(\alpha_1, \alpha_2, \alpha_2;\ \beta_1, \beta_2;\ \gamma_1, \gamma_2, \gamma_3;\ x, y, z)$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^r(\alpha_1)_s(\alpha_2)_r(\beta_1)_s(\beta_2)_{2r}}{(\gamma_2)_r(\gamma_3)_r(\gamma_1)_s}x^s y^r z^r$$

$$\times F^{(4)}[\beta_2+2r;\ \alpha_2+r;\ \gamma_2+r_1\gamma_3+r;\ y, z] \quad (13)$$

$\varphi_6(\alpha_1, \alpha_2, \alpha_3;\ \beta_1, \beta_2;\ \gamma_1, \gamma_2;\ x, y, z)$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_s(\alpha_2)_s(\alpha_3)_r(\beta_1)_s(\beta_2)_r}{(\gamma_1)_s(\gamma_2)_r\, r!\, s!}x^s y^r z^r$$

$$\times F^{(3)}[\beta_2+r,\ \beta_2+r;\ \alpha_2+r_1,\ \alpha_3+r;\ \gamma_2+2r;\ y, z] \quad (14)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^r(\alpha_1)_s(\alpha_2)_r(\alpha_3)_r(\beta_1)_s(\beta_2)_{2r}}{(\gamma_1)_s(\gamma_2)_{2r}(\gamma_2+r-1)_r\, r!\, s!}x^s y^r z^r$$

$$\times F^{(3)}[\beta_2+2r;\ \alpha_2+r.\ \alpha_3+r;\ \gamma_2+2r,\ \gamma_2+2r;\ y, z] \quad (15)$$

$$\varphi_7(\alpha_1, \alpha_1, \alpha_2;\ \beta_1, \beta_2;\ \gamma_1, \gamma_2;\ x, y, z)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^r(\alpha_1)_r(\alpha_2)_s(\beta_1)_r(\beta_2)_{r+s}}{(\gamma_1)_r(\gamma_2)_{r+s}\, r!\ s!}\, x^r y^s z^r$$

$$\times F^{(2)}[\alpha_1+r;\ \beta_1+r, \beta_2+r+s;\ \gamma_1+r, \gamma_2+r+s;\ x, z] \quad (16)$$

$$\varphi_8(\alpha_1, \alpha_2;\ \beta_1, \beta_2;\ \gamma_1, \gamma_2;\ x, y, z)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_s(\alpha_2)_{2r}(\beta_1)_s(\beta_2)_r}{(\gamma_1)_s(\gamma_2)_{2r}\, r!\ s!}\, x^s y^r z^r$$

$$\times F^{(1)}[\alpha_2+2r;\ \beta_2+r, \beta_2+r;\ \gamma_2+2r;\ y, z] \quad (17)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^r(\alpha_1)_s(\alpha_2)_{2r}(\beta_1)_s(\beta_2)_{2r}x^s y^r z^r}{(\gamma_1)_s(\gamma_2+r-1)_r(\gamma_2)_{2r}\, r!\ s!}$$

$$\times F^{(1)}[\alpha_2+2r, \beta_2+2r;\ \gamma_2+2r, \gamma_2+2r;\ y, z] \quad (18)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_s(\beta_1)_s(\alpha_2)_{2r}(\beta_2)_r(\gamma_2-\beta_2)_r}{(\gamma_1)_r(\gamma_2)_{2r}(\gamma_2+r-1)_r\, r!\ s!}\, x^s y^r z^r$$

$$\times F^{(2)}[\alpha_2+2r;\ \beta_2+r, \beta_2+r;\ \gamma_2+2r, \gamma_2+2r;\ y, z] \quad (19)$$

2. Relation between φ-function & simple Hyper-geometric function:

$$\varphi_4(\alpha_1, \alpha_2, \alpha_2;\ \beta_1, \gamma_1;\ x, y, z)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^s(\alpha_1)_r(\alpha_2)_s(\beta_1)_{r+2s}}{(\gamma_1)_{r+2s}\, r!\ s!}\, x^r y^s z^s$$

$$\times F(\beta_1+r+2s, \alpha_2+s;\ \gamma_1+r+2s;\ y+z) \quad (20)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_s(\beta_1)_{r+s}(\gamma_1-\beta_2)_s}{(\gamma_1)_{r+2s}\, r!\ s!}\, x^r y^s z^s$$

$$\times F(\beta_1+r+s, \alpha_2+s;\ \gamma_1+r+2s,\ y+z-yz) \quad (21)$$

$$\varphi_5(\alpha_1, \alpha_2, \alpha_3, \beta_1;\ \gamma_1, \gamma_2;\ x, y, z)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_s(\alpha_3)_r(\beta_1)_{r+s}}{(\gamma_1)_r(\gamma_2)_{r+s}\, r!\ s!}\, x^r y^s z^r$$

$$\times F(\beta_1+r+s, \alpha_1+r, \gamma_1+r;\ x)F(\beta_1+r+s, \alpha_3+r;\ \gamma_2+r+s;\ z) \quad (22)$$

$$\varphi_6(\alpha_1, \alpha_2, \alpha_3;\ \beta_1, \beta_2;\ \gamma_1, \gamma_2, \gamma_3;\ x, y, z)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_s(\alpha_2)_r(\alpha_3)_r(\beta_1)_s(\beta_2)_r}{(\gamma_1)_s(\gamma_2)_r(\gamma_3)_r\, r!\ s!}\, x^s y^r z^r$$

$$\times F(\beta_2+r, \alpha_2+r;\ \gamma_2+r;\ y)F(\beta_2+r, \alpha_3+r, \gamma_3+r;\ z) \quad (23)$$

$$\varphi_6(\alpha_1, \alpha_2, \alpha_2;\ \beta_1, \beta_2;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^r(\alpha_1)_s(\alpha_2)_r(\beta_1)_s(\beta_2)_{2r}(\gamma_2-\alpha_2)_r}{(\gamma_1)_s(\gamma_2)_r(\gamma_2)_{2r}\, r!\ s!}\, x^s y^r z^r$$

$$\times F(\beta_2+2r, \alpha_2+r;\ \gamma_2+2r;\ y+z) \quad (24)$$

$$\varphi_7(\alpha_1, \alpha_2, \alpha_3 ; \beta_1, \beta_2 ; \gamma_1, \gamma_2 ; x, y, z)$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha_1)_s(\beta_1)_s(\beta_2)_r(\alpha_2)_r(\alpha_3)_r(\gamma_2-\beta_3)_r}{(\gamma_2+r-1)_r(\gamma_2)_{2r}(\gamma_1)_s\, r!\, s!} x^s y^r z^r$$

$$\times F(\beta_3+r, \alpha_2+r ; \gamma_2+2r ; y) F(\beta_2+r, \alpha_3+r ; \gamma_2+2r ; z) \tag{25}$$

$$\varphi_7(\alpha_1, \alpha_2, \alpha_2 ; \beta_1, \beta_2 ; \gamma_1, \gamma_2 ; x, y, z)$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-)^r(\alpha_1)_s(\alpha_2)_r(\beta_1)_s(\beta_2)_{2r}}{(\gamma_1)_s(\gamma_2)_{2r}\, r!\, s!} x^s y^r z^r$$

$$\times F(\beta_2+2r, \alpha_2+r ; \gamma_2+2r ; y+z) \tag{26}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha_1)_s(\alpha_2)_r(\beta_1)_s(\beta_2)_r(\gamma_2-\beta_2)_r}{(\gamma_1)_s(\gamma_2)_{2r}\, r!\, s!} x^s y^r z^r$$

$$\times F(\beta_2+r, \alpha_2+r ; \gamma_2+2r ; y+z-yz) \tag{27}$$

$$\varphi_8(\alpha_1, \alpha_2 ; \beta_1, \beta_2 ; \gamma_1, \gamma_2 ; x, y, z)$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha_1)_s(\alpha_2)_r(\beta_1)_s(\beta_2)_r}{(\gamma_1)_s(\gamma_2)_r\, r!\, s!} x^s y^r z^r$$

$$\times F(\alpha_2+r, \beta_2+r ; \gamma_2+r ; y+z-yz) \tag{28}$$

$$\varphi_9(\alpha_1 ; \beta_1 ; \gamma_1 ; x, y, z)$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\beta_1)_r}{(\gamma_1)_r\, r!} x^r F(\alpha_1+r ; \beta_1+r ; \gamma_1+r ; y+z) \tag{29}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha_1)_{r+s}(\beta_1)_{r+s}}{(\gamma_1)_{r+s}\, r!\, s!} x^r y^s z^s$$

$$\times F(\alpha_1+r+s ; \beta_1+r+s ; \gamma_1+r+s ; y+z-yz) \tag{30}$$

3. **Partial differential Equations:**

Let A, B, C, L, M, N, P, S, R stand for

$$\frac{\partial^2 w}{dx^2}, \quad \frac{\partial^2 w}{\partial y^2}, \quad \frac{\partial^2 w}{\partial z^2}, \quad \frac{\partial^2 w}{\partial x \partial y}, \quad \frac{\partial^2 w}{\partial y \partial z}, \quad \frac{\partial^2 w}{\partial x \partial z}, \quad \frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial y}, \quad \frac{\partial w}{\partial z}$$

$$\varphi_1 \begin{bmatrix} x(1-x)A + \{\gamma_1-(\alpha_1+\beta_1+1)x\}P - \alpha_1\beta_1 w = 0 \\ y(1-y)B + \{\gamma_2-(\alpha_2+\beta_2+1)y\}Q - \alpha_2\beta_2 w = 0 \\ z(1-z)C + \{\gamma_3-(\alpha_3+\beta_3+1)z\}P - \alpha_3\beta_3 w = 0 \end{bmatrix} \tag{31}$$

$$\varphi_2 \begin{bmatrix} x(1-x)A + yL + zN + \{\gamma_1- \alpha_1+\beta_1+1)x\}P - \alpha_1\beta_1 w = 0 \\ y(1-y)B + zM + xL + \{\gamma_1-(\alpha_2+\beta_2+1)y\}Q - \alpha_2\beta_2 w = 0 \\ z(1-z)C + xN + yM + \}\gamma_1-(\alpha_3+\beta_3+1)z\}R - \alpha_3\beta_3 w = 0 \end{bmatrix} \tag{32}$$

$$\varphi_3 \begin{bmatrix} x(1-x)A + yL + \{\gamma_1-(\alpha_1+\beta_1+1)x\}P - \alpha_1\beta_1 w = 0 \\ y(1-y)B + zM + \{\gamma_2-(\alpha_2+\beta_2+1)y\}Q - \alpha_2\beta_2 w = 0 \\ z(1-z)C + xN + \{\gamma_2-(\alpha_3+\beta_3+1)z\}R - \alpha_3\beta_3 w = 0 \end{bmatrix} \tag{33}$$

$$\varphi_4\left[\begin{array}{l} x(1-x)A+y(1-x)L+\{\gamma_1-(\alpha_1+\beta_1+1)x\}P-\alpha_1y_1Q-\alpha_1zR+z(1-x)N-\alpha_1\beta_1w = 0 \\ y(1-y)B+x(1-y)L+z(1-y)+\{\gamma_1-(\alpha_2+\beta_1+1)y\}Q-\alpha_2zR-\alpha_2xP-\alpha_2\beta_1w = 0 \\ z(1-z)C+x(1-z)N+y(1-z)M+\{\gamma_1-(\alpha_3+\beta_1+1)z\}P-\alpha_3xP-\alpha_3\beta_1w = 0 \end{array}\right] \tag{34}$$

$$\varphi_5\left[\begin{array}{l} x(1-x)A-xyL-xzN+\{\gamma_1-(\alpha_1+\beta_1+1)x\}P-\alpha_1yQ-\alpha_1zR-\alpha_1\beta_1w = 0 \\ y(1-y)B-yzM-xyL+\{\gamma_2-(\alpha_3+\beta_1+1)y\}Q-\alpha_2xP-\alpha_2zR-\alpha_2\beta_1w = 0 \\ z(1-z)C-xzN-yzM+\{\gamma_2-(\alpha_3+\beta_1+1)z\}R-\alpha_3xP-\alpha_3yQ-\alpha_3\beta_1w = 0 \end{array}\right] \tag{35}$$

$$\varphi_6\left[\begin{array}{l} x(1-x)A+\{\gamma_1-(\alpha_1+\beta_1+1)x\}P-\alpha_1\beta_1w = 0 \\ y(1-y)B-yzM+\{\gamma_2-(\alpha_2+\beta_2+1)y\}Q-\alpha_2zR-\alpha_2\beta_2w = 0 \\ z(1-z)C-yzM+\{\gamma_3-(\alpha_3+\beta_2+1)z\}R-\alpha_3yQ-\alpha_3\beta_2w = 0 \end{array}\right] \tag{36}$$

$$\varphi_7\left[\begin{array}{l} x(1-x)A+\{\gamma_1-(\alpha_1+\beta_1+1)x\}P-\alpha_1\beta_1w = 0 \\ y(1-y)B+zc-yzM+\{\gamma_2-(\alpha_2+\beta_2+1)y\}Q-\alpha_2zR-\alpha_2\beta_2w = 0 \\ z(1-z)C+yB-yzM+\{\gamma_2-(\alpha_2+\beta_2+1)z\}R-\alpha_2yQ-\alpha_1\beta_2w = 0 \end{array}\right] \tag{37}$$

$$\varphi_8\left[\begin{array}{l} x(1-x)A+\{\gamma_1-(\alpha_1+\beta_1+1)x\}P-\alpha_1\beta_1w = 0 \\ y(1-y)B+z(1-z)C-2yzM+\{\gamma_2-(\alpha_2+\beta_2+1)y\}Q-(\alpha_2+\beta_2+1)zR-\alpha_2\beta_2w = 0 \\ z(1-z)C+y(1-y)B-2yzM+\{\gamma_2-(\alpha_2+\beta_2+\gamma)z\}R-(\alpha_2+\beta_2+1)yQ-\alpha_2\beta_2w = 0 \end{array}\right] \tag{38}$$

$$\varphi_9\left[\begin{array}{l} x(1-x)A+y(1-y)B+z(1-z)C-2xyL-2yzM-2xzN \\ \qquad -(\alpha_1+\beta_1+1)(xP+yQ+Rz)-\gamma_1P-\alpha_1\beta_1w = 0 \\ x(1-x)A+y(1-y)B+z(1-z)C-2xyL-2yzM-2xzN \\ \qquad -(\alpha_1+\beta_1+1)(xP+yQ+Rz)-\gamma_1Q-\alpha_1\beta_1w = 0 \\ x(1-x)A+y(1-y)B+z(1-z)C-2xyL-2yzM-2xzN \\ \qquad -(\alpha_1+\beta_1+1)(xP+yQ+Rz)-\gamma_1R-\alpha_1\beta_1w = 0 \end{array}\right]$$

4. We know that

$$\varphi_6(\alpha_1,\ \alpha_2,\ \alpha_3;\ \beta_1,\ \beta_2;\ \gamma_1,\ \gamma_2;\ x,\ y,\ z)$$

$$= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}{\Gamma(\gamma_1-\alpha_1)(\gamma_2-\alpha_2)\Gamma(\gamma_3-\alpha_3)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}$$

$$\times \int_0^1\int_0^1\int_0^1 u^{\alpha_1-1}(1-u)^{\gamma_1-\alpha_1-1}v^{\alpha_2-1}(1-v)^{\gamma_2-\alpha_2-1}w^{\alpha_3-1}(1-w)^{\gamma_3-\alpha_3-1}$$

$$\times (1-ux)^{-\beta_2}(1-yv-zw)^{-\beta_1}dudvdw.$$

Let us make the following substitutions

(1) $u = 1-t,\ v = 1-s,\ w = p$

(2) $u = 1-t,\ v = s,\ w = p$

(3) $u = t,\ v = 1-s,\ w = 1-p$

(4) $u = t,\ v = 1-s,\ w = p$

(5) $u = t,\ v = s,\ w = 1-p$

so that $\varphi_6(\alpha_1,\ \alpha_2,\ \alpha_3\,;\ \beta_1,\ \beta_2\,;\ \gamma_1,\ \gamma_2\,;\ x,\ y,\ z)$

$$= (1-x)^{-\beta_2}(1-y)^{-\beta_1}\varphi_6\left(\gamma_1-\alpha_1,\ \gamma_2-\alpha_2,\ \alpha_3\,;\ \beta_1,\ \beta_2\,;\ \gamma_1,\ \gamma_2\,;\ \frac{-x}{1-x},\ \frac{-y}{1-y},\ \frac{z}{1-y}\right)$$

$$= (1-x)^{-\beta_1}\varphi_6(\gamma_1-\alpha_1,\ \alpha_2,\ \alpha_3\,;\ \beta_1,\ \beta_2\,;\ \gamma_1,\ \gamma_2\,;\ \frac{-x}{1-x},\ y,\ z)$$

$$= (1-y-z)^{-\beta_1}\varphi_6\left(\alpha_1, \gamma_2-\alpha_2,\ \gamma_3-\alpha_3\,;\ \beta_1,\ \beta_2\,;\ \gamma_1,\ \gamma_2\,;\ x,\ -\frac{y}{1-y-z},\ \frac{-z}{1-y-z}\right)$$

$$= (1-y)^{-\beta_1}\varphi_6\left(\alpha_1,\ \gamma_2-\alpha_2,\ \alpha_3\,;\ \beta_1,\ \beta_2\,;\ \gamma_1,\ \gamma_2\,;\ x,\ -\frac{y}{1-y},\ \frac{z}{1-y}\right)$$

$$= (1-z)^{-\beta_1}\varphi_6\left(\alpha_1,\ \alpha_2,\ \gamma_3-\alpha_3\,;\ \beta_1,\ \beta_2\,;\ \gamma_1,\ \gamma_2\,;\ x,\ \frac{y}{1-z},\ \frac{-z}{1-z}\right) \tag{40}$$

5. Relation between φ_4 and φ_5 function.

$\varphi_4(\alpha_1,\ \alpha_2,\ \alpha_3\,;\ \beta_1,\ \gamma_1\,;\ x,\ y,\ z)$

$$= \frac{\Gamma(\gamma_1)}{\Gamma(\beta_1)\Gamma(\gamma_1-\beta_1)}\int_0^1 u^{\beta_1-1}(1-u)^{\gamma_1-\beta_1-1}(1-ux)^{-\alpha_1}(1-uy)^{-\alpha_2}(1-uz)^{-\alpha_3}\,du$$

Put $u = 1/t$, we have

$$= \frac{\Gamma(\gamma_1)}{\Gamma(\beta_1)\Gamma(\gamma_1-\beta_1)}\int_1^\infty t^{\alpha_1+\alpha_2+\alpha_3-\gamma_1}(t-1)^{\gamma_1-\beta_1-1}(t-x)^{-\alpha_1}(t-y)^{-\alpha_2}(t-z)^{-\alpha_3}\,dt$$

$$(t-y)^{-\alpha_2} = \left(\frac{x}{y}\right)^{\alpha_2} t^{-\alpha_2}\left\{\left(1-\frac{x}{t}\right)-\left(1-\frac{x}{y}\right)\right\}^{-\alpha_2}$$

and

$$(t-x)^{-\alpha_3} = \left(\frac{x}{z}\right)^{\alpha_3} t^{-\alpha_3}\left\{\left(1-\frac{x}{t}\right)-\left(1-\frac{x}{z}\right)\right\}^{-\alpha_3}$$

substituting we get

$$\varphi_4(\alpha_1, \alpha_2, \alpha_3 ; \beta_1, \gamma_1 ; x, y, z)$$

$$= \left(\frac{x}{y}\right)^{\alpha_2}\left(\frac{x}{z}\right)^{\alpha_3}\varphi_5\left(\beta_1, \alpha_2, \alpha_3 ; \alpha_1+\alpha_2+\alpha_3 ; \alpha_1+\alpha_2+\alpha_3, \gamma_1 ; x, 1-\frac{x}{y}, 1-\frac{x}{z}\right)$$

$$= \left(\frac{y}{x}\right)^{\alpha_1}\left(\frac{y}{z}\right)^{\alpha_3}\varphi_5\left(\alpha_1, \alpha_3, \beta_1 ; \alpha_1+\alpha_2+\alpha_3 ; \alpha_1+\alpha_2+\alpha_3 ; \gamma_1 ; 1-\frac{y}{x}, y, 1-\frac{y}{z}\right)$$

$$= \left(\frac{z}{x}\right)^{\alpha_1}\left(\frac{z}{y}\right)^{\alpha_2}\varphi_5\left(\alpha_1, \alpha_2, \beta_1, \alpha_1+\alpha_2+\alpha_3 ; \alpha_1+\alpha_2+\alpha_3, \gamma_1 ; 1-\frac{z}{x}, 1-\frac{z}{y}, z\right) \quad (41)$$

In conclusion, I wish to express my indebtedness to Dr. S. C. Mitra who kindly suggested the problem to me and took keen interest in the preparation of this paper.

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VERSUCH EINER *LOGISCHEN* QUANTENDYNAMIK DES ELEKTRONS. (II)

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Inhaltsübersicht. Es erfolgen unten in Fortsetzung des ersten Teils unserer unter dem gleichen Titel erschienenen Arbeit die Anwendungen der dort vorgebrachten Theorie auf einige wichtige Beispiele.

1. Vorbemerkungen

Im Anschluss am ersten Teil unserer unter dem gleichen Titel erschienenen Arbeit wollen wir im Nachstehenden die Anwendungen der dort vorgelegten Theorie auf (Kar, 1952) einige wichtige Beispiele behandeln. Als dem wichtigsten Beispiel wenden wir uns zunächst dem Fall des Elektrons im atomaren Felde zu. Danach wollen wir nacheinander noch folgende Fälle in Betracht ziehen, nämlich, den Fall des Elektrons im konstanten elektrischen Felde, den Fall des Elektrons im konstanten magnetischen Felde und den Fall des feldfreien Elektrons. Jedes Beispiel wird auf zwei Weise behandelt, nämlich, nach den quantendynamischen Gleichungen $[B_2, A_2]$ bzw. $[B_2(a)], A_2(a)]$ des ersten Teils und möglichst unter den Rubriken: (*i*) die unverkürzte Quantenbahn, (*ii*) die klassische Bahn u. (*iii*) die Quantenbahn bei nullter Näherung bzw. bis auf die erste Näherung.

Die Numerierung der im vorliegenden Teile (II) auftretenden Gleichungen bzw. Sätze ist als unabhängig von der im Teile (I) geübten zu verstehen. Nötigenfalls geschieht ein Verweis auf irgend welchen Satz des Teils (I) explizite.

2. Das Elektron im atomaren Felde

Aus (I) holen wir die Gleichungen (26) bzw. (26(*a*)) her. Mit der Setzung

$$\varphi = \frac{Ze}{r} = V, \; A_x = 0 = A_y = A_z$$

lauten sie

$$\left.\begin{aligned} \left[\left(h_0\frac{\partial}{\partial t} + eV\right) + ch_0(\vec{\sigma}\vec{\nabla})\right]\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix} &= +\varepsilon_0\begin{vmatrix}\psi_1'\\ \psi_2'\end{vmatrix}, \\ \left[\left(h_0\frac{\partial}{\partial t} + eV\right) - ch_0(\vec{\sigma}\vec{\nabla})\right]\begin{vmatrix}\psi_1'\\ \psi_2'\end{vmatrix} &= +\varepsilon_0\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}, \end{aligned}\right\} \tag{1}$$

bzw.

$$\left.\begin{aligned} \left[\left(h_0\frac{\partial}{\partial t} - eV\right) + ch_0(\vec{\sigma}\vec{\nabla})\right]\begin{vmatrix}\chi_1\\ \chi_2\end{vmatrix} &= -\varepsilon_0\begin{vmatrix}\chi_1'\\ \chi_2'\end{vmatrix}, \\ \left[\left(h_0\frac{\partial}{\partial t} - eV\right) - ch_0(\vec{\sigma}\vec{\nabla})\right]\begin{vmatrix}\chi_1'\\ \chi_2'\end{vmatrix} &= -\varepsilon_0\begin{vmatrix}\chi_1\\ \chi_2\end{vmatrix}, \end{aligned}\right\} \tag{2}$$

(a) *Einige transformationstheoretische Sätze.* Wir wollen nun einige bekannte zur Transformation auf Polarkoordinaten nützliche Sätze vorausschicken.

Mit den Bezeichnungen

$$\left.\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi, \\ z &= r \cos\theta \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} L_x &= \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \\ L_y &= \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \\ L_z &= \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial\varphi}, \end{aligned}\right\} \tag{4}$$

$$\vec{M} = \vec{L} + \frac{i}{2}\vec{\sigma}, \tag{5}$$

beweist man leicht den Satz

$$\left.\begin{aligned} M_z\left[\left(h_0\frac{\partial}{\partial t} + eV\right) + ch_0(\vec{\sigma}\vec{\nabla})\right] &= \left[\left(h_0\frac{\partial}{\partial t} + eV\right) + ch_0(\vec{\sigma}\vec{\nabla})\right]M_z, \\ M_z\left[\left(h_0\frac{\partial}{\partial t} + eV\right) - ch_0(\vec{\sigma}\vec{\nabla})\right] &= \left[\left(h_0\frac{\partial}{\partial t} + eV\right) - ch_0(\vec{\sigma}\vec{\nabla})\right]M_z. \end{aligned}\right\} \tag{6}$$

Mit der Bezeichnung: $(\vec{\sigma}\vec{\nabla}) + i = K$, beweist man ferner den Satz

$$\left.\begin{aligned} K\left[\left(h_0\frac{\partial}{\partial t} + eV\right) + ch_0(\vec{\sigma}\vec{\nabla})\right] &= \left[\left(h_0\frac{\partial}{\partial t} + eV\right) - ch_0(\vec{\sigma}\vec{\nabla})\right]K, \\ K\left[\left(h_0\frac{\partial}{\partial t} + eV\right) - ch_0(\vec{\sigma}\vec{\nabla})\right] &= \left[\left(h_0\frac{\partial}{\partial t} + eV\right) + ch_0(\vec{\sigma}\vec{\nabla})\right]K, \end{aligned}\right\} \tag{7}$$

Diese zunächst für die Gleichungen (1) vorgesehene Sätze gelten offenbar auch fur das Gleichungspaar (2),

Ebenso beweist man allgemein die Sätze

$$M_z.\,K = K.\,M_z, \tag{8}$$

$$M^2 = K^2 + \tfrac{1}{4}, \tag{9}$$

$$\left(\vec{\sigma}\frac{\vec{r}}{r}\right)^2 = 1, \tag{10}$$

$$\left(\vec{\sigma}\frac{\vec{r}}{r}\right)K + K.\left(\vec{\sigma}\frac{\vec{r}}{r}\right) = 0, \tag{11}$$

$$(\vec{\sigma}\vec{\nabla}) = \left(\vec{\sigma}\frac{\vec{r}}{r}\right).\frac{d}{dr} + \frac{i}{r}\left(\vec{\sigma}\frac{\vec{r}}{r}\right)(\vec{\sigma}\vec{L}) \tag{12}$$

(*b*) *Schrittweise Transformation des Gleichungspaares* (1) *bzw.* (2) *auf Polarkoordinaten.*

Auf Grund des Satzes (6) setzen wir

$$\left.\begin{aligned} \left(L_z+\frac{i}{2}\sigma_z\right)\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix} &= i\mu\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}, \\ \text{bzw}\left(L_z+\frac{i}{2}\sigma_z\right)\begin{vmatrix}\psi_1'\\ \psi_2'\end{vmatrix} &= i\mu\begin{vmatrix}\psi_1'\\ \psi_2'\end{vmatrix}, \end{aligned}\right\} \tag{13}$$

wo μ eine Konstante ist. Wir erwägen nun die zwei Möglichkeiten:

$$(a)\quad \mu=+(m+\tfrac{1}{2}),$$

$$(b)\quad \mu=-(m+\tfrac{1}{2}),$$

wo m eine ganze Zahl sei.

Entsprechend der Wahl (a) findet man leicht

$$(\psi_1,\psi_1')\sim e^{im\varphi},\ (\psi_2,\psi_2')\sim e^{i(m+1)\varphi}, \tag{14}$$

entsprechend aber der Wahl (b)

$$(\psi_1,\psi_1')\sim e^{-i(m+1)\varphi},\ (\psi_2,\psi_2')\sim e^{-im\varphi}. \tag{15}$$

Auf Grund des Satzes (7) setzen wir

$$\left.\begin{aligned} K\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix} &= \{(\vec{\sigma}\vec{L})+i\}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}=ik\begin{vmatrix}\psi_1'\\ \psi_2'\end{vmatrix}, \\ K\begin{vmatrix}\psi_1'\\ \psi_2'\end{vmatrix} &= \{(\vec{\sigma}\vec{L})+i\}\begin{vmatrix}\psi_1'\\ \psi_2'\end{vmatrix}=ik\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}, \end{aligned}\right\} \tag{16}$$

wo k eine Konstante ist. Wir führen nun die Bezeichnungen

$$\left.\begin{aligned} \tfrac{1}{2}(\psi_1+\psi_1') &= f_1,\ \tfrac{1}{2}(\psi_1-\psi_1')=g_1, \\ \tfrac{1}{2}(\psi_2+\psi_2') &= f_2,\ \tfrac{1}{2}(\psi_2-\psi_2')=g_2, \end{aligned}\right\} \tag{17}$$

ein und erhalten

$$\left.\begin{aligned} [(\vec{\sigma}\vec{L})+i]\begin{vmatrix}f_1\\ f_2\end{vmatrix} &= ik\begin{vmatrix}f_1\\ f_2\end{vmatrix}, \\ [(\vec{\sigma}\vec{L})+i]\begin{vmatrix}g_1\\ g_2\end{vmatrix} &= -ik\begin{vmatrix}g_1\\ g_2\end{vmatrix} \end{aligned}\right\} \tag{18}$$

Hieraus ergeben sich die Gleichungspaare

$$\left.\begin{aligned} e^{i\varphi}\left\{\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\varphi}\right\}f_1 &= \{k+(\mu-\tfrac{1}{2})\}f_2 \\ -e^{-i\varphi}\left\{\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\varphi}\right\}f_2 &= \{k-(\mu+\tfrac{1}{2})\}f_1 \end{aligned}\right\} \tag{19}$$

und

$$\left.\begin{aligned} e^{i\varphi}\left\{\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\varphi}\right\}g_1 &= -\{k-(\mu-\tfrac{1}{2})\}\,g_2 \\ -e^{-i\varphi}\left\{\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\varphi}\right\}g_2 &= -\{k+(\mu+\tfrac{1}{2})\}\,g_1 \end{aligned}\right\} \tag{20}$$

Entsprechend der Wahl (*a*) oben findet man die Gleichnugspaare zweiter Ordnung wie

$$\left.\begin{aligned} \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+k(k-1)-\frac{m^2}{\sin^2\theta}\right]f_1 &= 0, \\ \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+k(k-1)-\frac{(m+1)^2}{\sin^2\theta}\right]f_2 &= 0, \end{aligned}\right\} \tag{21}$$

und

$$\left.\begin{aligned} \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+k(k+1)-\frac{m^2}{\sin^2\theta}\right]g_1 &= 0, \\ \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+k(k+1)-\frac{(m+1)^2}{\sin^2\theta}\right]g_2 &= 0, \end{aligned}\right\} \tag{22}$$

entsprechend aber der Wahl (b), die Gleichungspaare

$$\left.\begin{aligned} \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+k(k-1)-\frac{(m+1)^2}{\sin^2\theta}\right]f_1 &= 0, \\ \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+k(k-1)-\frac{m^2}{\sin^2\theta}\right]f_2 &= 0, \end{aligned}\right\} \tag{23}$$

und

$$\left.\begin{aligned} \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+k(k+1)-\frac{(m+1)^2}{\sin^2\theta}\right]g_1 &= 0 \\ \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+k(k+1)-\frac{m^2}{\sin^2\theta}\right]g_2 &= 0 \end{aligned}\right\} \tag{24}$$

Man hat also nun auch noch zwei Möglichkeiten in Erwägung zu ziehen, nämlich, entsprechend (α) $k=+(l+1)$, (β) $k=-(l+1)$, wo l eine positive ganze Zahl sei. Im ganzen hat man demnach die vier Fälle zu berücksichtigen, nämlich, (a, α), (b, α), (a, β), (b, β).

Fall: (a, α).

Auf Grund von (21) bzw. (22) kann man einstweilen setzen

$$\left.\begin{aligned} (f_1, g_1) &= [A(r,t)\,P_l^{-m},\ C(r,t)\,P_{l+1}^{-m}]\,e^{im\varphi}, \\ (f_2, g_2) &= [B(r,t)\,P_l^{-(m+1)},\ D(r,t)\,P_{l+1}^{-(m+1)}]\,e^{i(m+1)\varphi}, \end{aligned}\right\} \tag{25}$$

wo P_l^{-m}, $P_l^{-(m+1)}$, usw. die bekannten Legendrefunktionen und $A(r, t)$, $B(r, t)$, usw. noch zu bestimmen sind.

In Ansehung aber von (19) bzw. (20) mit Rücksicht auf Rekursionsgleichungen der Art

$$\left\{\frac{\partial}{\partial\theta}+(m+1)\cot\theta\right\}P_l^{-(m+1)}=-P_l^{-m}, \tag{26}$$

findet man leicht

$$\left.\begin{aligned} B(r,\, t) &= A(r,\, t)\,.\,(l-m),\\ D(r,\, t) &= -C(r,\, t)\,.\,(l+m+2). \end{aligned}\right\} \tag{27}$$

Fall: $(b,\ \alpha)$.

Setzt man auf Grund von (23) bzw. (24) zunächst

$$\left.\begin{aligned} (f_1,\, g_1) &= \left[A(r,\, t)\, P_l^{-(m+1)},\ C(r,\, t)\, P_{l+1}^{-(m+1)}\right] e^{-i(m+1)\varphi},\\ (f_2,\, g_2) &= \left[B(r,\, t)\, P_l^{-m},\ D(r,\, t)\, P_{l+1}^{-m}\right] e^{-im\varphi}, \end{aligned}\right\} \tag{28}$$

so ergibt sich nach (19) bzw. (20) mit Rücksicht auf (26)

$$\left.\begin{aligned} A(r,\, t) &= -B(r,\, t)\,.\,(l-m)\\ C(r,\, t) &= D(r,\, t)\,.\,(l+m+2) \end{aligned}\right\} \tag{29}$$

Fall: $(a,\ \beta)$.

Ähnliches Verfahren ergibt für diesen Fall

$$\left.\begin{aligned} (f_1.\, g_1) &= \left[A(r,\, t)\, P_{l+1}^{-m},\ C(r,\, t)\, P_l^{-m}\right] e^{im\varphi}\\ (f_2,\, g_2) &= \left[-A(r,\, t)\,.\,(l+m+2)\, P_{l+1}^{-(m+1)},\ C(r,\, t)\,.\,(l-m)\, P_l^{-(m+1)}\right] e^{i(m+1)\varphi} \end{aligned}\right\} \tag{30}$$

und für den

Fall: $(b,\ \beta)$

$$\left.\begin{aligned} (f_1,\, g_1) &= \left[B(r,\, t)\,.\,(l+m+2)\, P_{l+1}^{-(m+1)},\ -D(r,\, t)\,.\,(l-m)\, P_l^{-(m+1)}\right] e^{-i(m+1)\varphi}\\ (f_2,\, g_2) &= \left[B(r,\, t)\, P_{l+1}^{-m},\ D(r,\, t)\, P_l^{-m}\right] e^{-im\varphi} \end{aligned}\right\} \tag{31}$$

Mit Hilfe von (12) schreiben wir nun die Gleichungen (1), wie folgt, hin

$$\left[\left(h_0\frac{\partial}{\partial t}+eV\right)+ch_0\left\{\left(\vec{\sigma}\,\frac{\vec{r}}{r}\right)\frac{d}{dr}+\frac{i}{r}\left(\vec{\sigma}\,\frac{\vec{r}}{r}\right)\left(\vec{\sigma}\vec{L}\right)\right\}\right]\left|\begin{matrix}\psi_1\\ \psi_2\end{matrix}\right|=+\varepsilon_0\left|\begin{matrix}\psi_1'\\ \psi_2'\end{matrix}\right|,$$

$$\left[\left(h_0\frac{\partial}{\partial t}+eV\right)-ch_0\left\{\left(\vec{\sigma}\,\frac{\vec{r}}{r}\right)\frac{d}{dr}+\frac{i}{r}\left(\vec{\sigma}\,\frac{\vec{r}}{r}\right)\left(\vec{\sigma}\vec{L}\right)\right\}\right]\left|\begin{matrix}\psi_1'\\ \psi_2'\end{matrix}\right|=+\varepsilon_0\left|\begin{matrix}\psi_1\\ \psi_2\end{matrix}\right|,$$

Daraus folgen mit Hilfe von (16)

$$\left.\begin{aligned} \left[\left(h_0\frac{\partial}{\partial t}+eV\right)+ch_0\left(\vec{\sigma}\,\frac{\vec{r}}{r}\right)\left\{\frac{d}{dr}+\frac{1}{r}\right\}\right]\left|\begin{matrix}\psi_1\\ \psi_2\end{matrix}\right| &= \left[\frac{ch_0 k}{r}\left(\vec{\sigma}\frac{\vec{r}}{r}\right)+\varepsilon_0\right]\left|\begin{matrix}\psi_1'\\ \psi_2'\end{matrix}\right|,\\ \left[\left(h_0\frac{\partial}{\partial t}+eV\right)-ch_0\left(\vec{\sigma}\,\frac{\vec{r}}{r}\right)\left\{\frac{d}{dr}+\frac{1}{r}\right\}\right]\left|\begin{matrix}\psi_1'\\ \psi_2'\end{matrix}\right| &= \left[-\frac{ch_0 k}{r}\left(\vec{\sigma}\,\frac{\vec{r}}{r}\right)+\varepsilon_0\right]\left|\begin{matrix}\psi_1\\ \psi_2\end{matrix}\right|, \end{aligned}\right\} \tag{32}$$

daraufhin aber mittles (17)

$$\left.\begin{aligned} &\left(h_0\frac{\partial}{\partial t}+eV-\varepsilon_0\right)\begin{vmatrix} rf_1 \\ rf_2 \end{vmatrix}+ch_0\left(\frac{d}{dr}+\frac{k}{r}\right)\left(\vec{\sigma}\frac{\vec{r}}{r}\right)\begin{vmatrix} rg_1 \\ rg_2 \end{vmatrix}=0, \\ &\left(h_0\frac{\partial}{\partial t}+eV+\varepsilon_0\right)\begin{vmatrix} rg_1 \\ rg_2 \end{vmatrix}+ch_0\left(\frac{d}{dr}-\frac{k}{r}\right)\left(\vec{\sigma}\frac{\vec{r}}{r}\right)\begin{vmatrix} rf_1 \\ rf_2 \end{vmatrix}=0. \end{aligned}\right\} \tag{33}$$

Blickt man auf den Fall (a, α) zuruck und schreibt man statt (25)

$$(f_1, f_2) = A(r, t)\left[P_l^{-m}e^{im\varphi},\ (l-m)\,P_l^{-(m+1)}e^{i(m+1)\varphi}\right]$$

$$(g_1, g_2) = C(r, t)\left[P_{l+1}^{-m}e^{im\varphi},\ -(l+m+2)\,P_{l+1}^{-(m+1)}e^{i(m+1)\varphi}\right]$$

so hat man

$$\left(\vec{\sigma}\frac{\vec{r}}{r}\right)\begin{vmatrix} rg_1 \\ rg_2 \end{vmatrix}=\begin{vmatrix} \cos\theta, & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi}, & -\cos\theta \end{vmatrix} r.C(r, t)\begin{vmatrix} P_{l+1}^{-m}e^{im\varphi} \\ -(l+m+2)P_{l+1}^{-(m+1)}e^{i(m+1)\varphi} \end{vmatrix}$$

$$= -rC(r,t)\begin{vmatrix} P_l^{-m}e^{im\varphi} \\ (l-m)P_l^{-(m+1)}e^{im+1)\varphi} \end{vmatrix} \tag{34}$$

mittels der Rekursionsgleichungen

$$\left.\begin{aligned} &\cos\theta\,P_{l+1}^{-m}-(l+m+2)\sin\theta\,P_{l+1}^{-(m+1)}=-P_l^{-m}, \\ &\sin\theta\,P_{l+1}^{-m}+(l+m+2)\cos\theta\,P_{l+1}^{-(m+1)}=-(l-m)P_l^{-(m+1)}. \end{aligned}\right\} \tag{35}$$

Danach lauten die radialen Gleichungen für diesen Fall

$$\left.\begin{aligned} &\left(h_0\frac{\partial}{\partial t}+eV-\varepsilon_0\right)(rA)-ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rC)=0, \\ &\left(h_0\frac{\partial}{\partial t}+eV+\varepsilon_0\right)(rC)-ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rA)=0. \end{aligned}\right\} \tag{36}$$

Analoges Verfahren liefert für den *Fall*: (b, α)

$$\left.\begin{aligned} &\left(h_0\frac{\partial}{\partial t}+eV-\varepsilon_0\right)(rB)+ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rD)=0, \\ &\left(h_0\frac{\partial}{\partial t}+eV+\varepsilon_0\right)(rD)+ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rB)=0. \end{aligned}\right\} \tag{37}$$

In gleicher Weise findet man fur die Fälle: (a, β) bzw. (b, β) die radialen Gleichungen als

$$\left.\begin{aligned} &\left(h_0\frac{\partial}{\partial t}+eV-\varepsilon_0\right)(rA)-ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rC)=0, \\ &\left(h_0\frac{\partial}{\partial t}+eV+\varepsilon_0\right)(rC)-ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rA)=0, \end{aligned}\right\} \tag{38}$$

bzw.

$$\left.\begin{aligned}\left(h_0\frac{\partial}{\partial t}+eV-\varepsilon_0\right)(rB)+ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rD) &= 0,\\ \left(h_0\frac{\partial}{\partial t}+eV+\varepsilon_0\right)(rD)+ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rB) &= 0.\end{aligned}\right\} \tag{39}$$

Die Transformation der Gleichungen (1) auf Polarkoordinaten ist somit beendet. Will man die gleiche Transformation der Gleichungen (2) ausgeführt haben und führt man an Stelle von (17) die Bezeichnungen:

$$\left.\begin{aligned}\tfrac{1}{2}(X_1+X_1') = f_1,\ \ \tfrac{1}{2}(X_1-X_1') = g_1\\ \tfrac{1}{2}(X_2+X_2') = f_2,\ \ \tfrac{1}{2}(X_2-X_2') = g_2\end{aligned}\right\} \tag{40}$$

ein, so findet man die entsprechenden radialen Gleichungen von gleicher Gestalt wie oben mit dem einzigen Unterschied, dass durchweg e bzw. ε_0 durch $-e$ bzw. $-\varepsilon_0$ ersetzt werden.

(c) *Abspaltung des Zeitfaktors und Normierung der radialen Gleichungen..*

In den Gleichungen (1) setzen wir den Zeitfaktor zu

$$\exp\left(-\frac{\varepsilon_n}{h_0}\right) = \underset{-}{Z}\ , \tag{41}$$

in den Gleichungen (2) aber, zu

$$\exp\left(+\frac{\varepsilon_n}{h_0}\right) = \underset{+}{Z}. \tag{42}$$

Fall: (a, α). Aus (36) hat man also nach Abspaltung von $\underset{-}{Z}$

$$\left.\begin{aligned}\left(-\varepsilon_n+\frac{Ze^2}{r}-\varepsilon_0\right)(rA) &= ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rC),\\ \left(-\varepsilon_n+\frac{Ze^2}{r}+\varepsilon_0\right)(rC) &= ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rA),\end{aligned}\right\} \tag{43}$$

nach Abspaltung von $\underset{+}{Z}$ aber das Paar

$$\left.\begin{aligned}\left(\varepsilon_n-\frac{Ze^2}{r}+\varepsilon_0\right)(rA) &= ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rC),\\ \left(\varepsilon_n-\frac{Ze^2}{r}-\varepsilon_0\right)(rC) &= ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rA).\end{aligned}\right\} \tag{44}$$

Mit der Setzung:

$$\left.\begin{aligned}rA &= U(r),\\ rC &= V(r),\end{aligned}\right\} \tag{45}$$

wird nun (43) auf die Form

$$\left.\begin{aligned}\left\{\left(-\varepsilon_n+\frac{Ze^2}{r}\right)-\varepsilon_0\right\}U &= ch_0\left\{\frac{d}{dr}+\frac{l+1}{r}\right\}V,\\ \left\{\left(-\varepsilon_n+\frac{Ze^2}{r}\right)+\varepsilon_0\right\}V &= ch_0\left\{\frac{d}{dr}-\frac{l+1}{r}\right\}U.\end{aligned}\right\}\tag{A}$$

normiet. Man bringt (44) auf die gleiche Form hin mit der Setzung

$$\left.\begin{aligned}rA &= U(r),\\ rC &= -V(r),\end{aligned}\right\}\tag{46}$$

Fall: (b, α). Aus (37) hat man nach Abspaltung von $\underset{-}{Z}$

$$\left.\begin{aligned}\left(-\varepsilon_n+\frac{Ze^2}{r}-\varepsilon_0\right)(rB) &= -ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rD),\\ \left(-\varepsilon_n+\frac{Ze^2}{r}+\varepsilon_0\right)(rD) &= -ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rB).\end{aligned}\right\}\tag{47}$$

von $\underset{+}{Z}$ aber

$$\left.\begin{aligned}\left(\varepsilon_n-\frac{Ze^2}{r}+\varepsilon_0\right)(rB) &= -ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rD),\\ \left(\varepsilon_n-\frac{Ze^2}{r}-\varepsilon_0\right)(rD) &= -ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rB).\end{aligned}\right\}\tag{48}$$

Mit der Setzung

$$\left.\begin{aligned}rB &= U(r),\\ rD &= -V(r),\end{aligned}\right\}\tag{49}$$

in (47), aber

$$\left.\begin{aligned}rB &= U(r),\\ rD &= V(r),\end{aligned}\right\}\tag{50}$$

in (48) bringt man beide Gleichungspaare auf die Form (A) hin.

Fall: (a, β). Aus (38) folgt nach Abspaltung von $\underset{-}{Z}$

$$\left.\begin{aligned}\left(-\varepsilon_n+\frac{Ze^2}{r}-\varepsilon_0\right)(rA) &= ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rC),\\ \left(-\varepsilon_n+\frac{Ze^2}{r}+\varepsilon_0\right)(rC) &= ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rA),\end{aligned}\right\}\tag{51}$$

von $\underset{+}{Z}$ aber

$$\left.\begin{aligned}\left(\varepsilon_n-\frac{Ze^2}{r}+\varepsilon_0\right)(rA) &= ch_0\left(\frac{d}{dr}-\frac{l+1}{r}\right)(rC),\\ \left(\varepsilon_n-\frac{Ze^2}{r}-\varepsilon_0\right)(rC) &= ch_0\left(\frac{d}{dr}+\frac{l+1}{r}\right)(rA).\end{aligned}\right\}\tag{52}$$

Setzt man nun

$$\left.\begin{aligned} rA &= v(r), \\ rC &= u(r), \end{aligned}\right\} \tag{53}$$

in (51), aber

$$\left.\begin{aligned} rA &= v(r), \\ rC &= -u(r), \end{aligned}\right\} \tag{54}$$

in (52), so normiert man die beiden Gleichungspaare auf die Form

$$\left.\begin{aligned} \left\{\left(-\varepsilon_n + \frac{Ze^2}{r}\right) - \varepsilon_0\right\} v &= ch_0\left\{\frac{d}{dr} - \frac{l+1}{r}\right\} u, \\ \left\{\left(-\varepsilon_n + \frac{Ze^2}{r}\right) + \varepsilon_0\right\} u &= ch_0\left\{\frac{d}{dr} - \frac{l+1}{r}\right\} v, \end{aligned}\right\} \tag{B}$$

hin.

Fall: (b, β). Aus (39) folgt nach Abspaltung von $\underset{-}{Z}$

$$\left.\begin{aligned} \left(-\varepsilon_n + \frac{Ze^2}{r} - \varepsilon_0\right)(rB) &= -ch_0\left(\frac{d}{dr} - \frac{l+1}{r}\right)(rD), \\ \left(-\varepsilon_n + \frac{Ze^2}{r} + \varepsilon_0\right)(rD) &= -ch_0\left(\frac{d}{dr} + \frac{l+1}{r}\right)(rB), \end{aligned}\right\} \tag{55}$$

von $\underset{+}{Z}$ aber

$$\left.\begin{aligned} \left(\varepsilon_n - \frac{Ze^2}{r} + \varepsilon_0\right)(rB) &= -ch_0\left(\frac{d}{dr} - \frac{l+1}{r}\right)(rD), \\ \left(\varepsilon_n - \frac{Ze^2}{r} - \varepsilon_0\right)(rD) &= -ch_0\left(\frac{d}{dr} + \frac{l+1}{r}\right)(rB), \end{aligned}\right\} \tag{56}$$

Setzt man nun

$$\left.\begin{aligned} rB &= v(r), \\ rD &= -u(r), \end{aligned}\right\} \tag{57}$$

in (55), aber

$$\left.\begin{aligned} rB &= v(r), \\ rD &= u(r) \end{aligned}\right\} \tag{58}$$

in (56), so ergibt sich für die beiden Gleichungspaare die normierte Form (B) oben.

(*d*) *Lösung der normierten Gleichungspaare* (A) *bzw.* (B).

Das Gleichungspaar (A)

Überlegt man das Verhalten der Funktionen U bzw. V im Unendlichen, so kann man ansetzen

$$\left.\begin{aligned} U &= e^{-\alpha r}\sum_{\nu=1}^{\nu=s} B_\nu r^{\mu+\nu}, \\ V &= e^{-\alpha r}\sum_{\nu=1}^{\nu=s} C_\nu r^{\mu+\nu}, \end{aligned}\right\} \tag{59}$$

wobei

$$\alpha = \frac{1}{ch_0} \sqrt{(\varepsilon_n^2 - \varepsilon_0^2)}. \tag{60}$$

Das Einsetzen dieser Ausdrücke in (A) liefert

$$\left.\begin{aligned} -(\varepsilon_n - \varepsilon_n)\sum_{\nu=1}^{s} C_\nu r^{\mu+\nu} + Ze^2 \sum_{\nu=1}^{s} C_\nu r^{\mu+\nu-1} &= ch_0\Big[-\alpha \sum_{\nu=1}^{s} B_\nu r^{\mu+\nu} \\ &\quad + \sum_{\nu=1}^{s} B_\nu(\mu+\nu) r^{\mu+\nu-1} - (l+1)\sum_{\nu=1}^{s} B_\nu r^{\mu+\nu-1}\Big], \\ -(\varepsilon_n + \varepsilon_0)\sum_{\nu=1}^{s} B_\nu r^{\mu+\nu} + Ze^2 \sum_{\nu=1}^{s} B_\nu r^{\mu+\nu-1} &= ch_0\Big[-\alpha \sum_{\nu=1}^{s} C_\nu r^{\mu+\nu} \\ &\quad + \sum_{\nu=1}^{s} C_\nu(\mu+\nu) r^{\mu+\nu-1} + (l+1)\sum_{\nu=1}^{s} C_\nu r^{\mu+\nu-1}\Big]. \end{aligned}\right\} \tag{61}$$

Das Zusammenbestehen dieser beiden Koeffizientengleichungen erfordert den Ansatz

$$\left.\begin{aligned} B_{\nu+1} &= D_{\nu+1}[\{(\mu+\nu+1)+(l+1)\}\sqrt{(\varepsilon_n - \varepsilon_0)} - \gamma Z \sqrt{(\varepsilon_n + \varepsilon_0)}], \\ C_{\nu+1} &= D_{\nu+1}[\gamma Z \sqrt{(\varepsilon_n - \varepsilon_0)} - \{(\mu+\nu+1)-(l+1)\}\sqrt{(\varepsilon_n + \varepsilon_0)}], \end{aligned}\right\} \tag{62}$$

wobei

$$\gamma = \frac{e^2}{ch_0}. \tag{63}$$

Daher ergibt sich aus (61) die Rekursionsgleichung:

$$2[-\gamma Z . \varepsilon_n + \sqrt{(\varepsilon_n^2 - \varepsilon_0^2)}. \ (\mu+\nu)] . D_\nu = ch_0[-\gamma^2 Z^2 + \{(\mu+\nu+1)^2 - (l+1)^2]. D_{\nu+1}. \tag{64}$$

Aus der Bedingung: $B_0 = 0 = C_0 = D_0$, folgt

$$-\gamma^2 Z^2 + (\mu+1)^2 - (l+1)^2 = 0,$$

oder

$$\mu = -1 + \sqrt{\{(l+1)^2 + \gamma^2 Z^2\}}. \tag{65}$$

Aus $B_{s+1} = 0 = C_{s+1} = D_{s+1}$, folgt aber

$$-\gamma Z . \varepsilon_n + \sqrt{(\varepsilon_n^2 - \varepsilon_0^2)} . (\mu+s) = 0,$$

oder

$$\varepsilon_n = \varepsilon_0 \left\{1 - \frac{\gamma^2 Z^2}{(\mu+s)^2}\right\}^{-\frac{1}{2}}. \tag{66}$$

Das Gleichungspaar (B).

In Ansehung des Verhaltens von u, v im Unendlichen macht man auch hier die Ansätze

$$\left.\begin{aligned} u &= e^{-\alpha r} \sum_{\nu=1}^{\nu=s} b_\nu r^{\mu+\nu}, \\ v &= e^{-\alpha r} \sum_{\nu=1}^{\nu=s} c_\nu r^{\mu+\nu}, \end{aligned}\right\} \tag{67}$$

wobei, wie im vorigen Falle,

$$\alpha = \frac{1}{ch_0}\sqrt{(\varepsilon_n^2-\varepsilon_o^2)}$$

Nach Einsetzen dieser Ausdrucke in (B) hat man zunächst Gleichungen vom gleichen Aussehen wie (61) und dann die Koeffizientengleichungen

$$\left.\begin{aligned}(\varepsilon_n-\varepsilon_o)b_\nu-ch_0\alpha.c_\nu &= Ze^2.b_{\nu+1}-ch_0\{(\mu+\nu+1)+(l+1)\}c_{\nu+1},\\ (\varepsilon_n+\varepsilon_o)c_\nu-ch_0\alpha.b_\nu &= Ze^2.c_{\nu+1}-ch_0\{(\mu+\nu+1)-(l+1)\}b_{\nu+1}.\end{aligned}\right\} \tag{68}$$

Das Zusammenklingen dieser beiden erfordert den Ansatz

$$\left.\begin{aligned}b_{\nu+1} &= d_{\nu+1}[\{(\mu+\nu+1)+(l+1)\}\sqrt{(\varepsilon_n+\varepsilon_o-)}\gamma Z\sqrt{(\varepsilon_n-\varepsilon_o)}],\\ c_{\nu+1} &= d_{\nu+1}[\gamma Z\sqrt{(\varepsilon_n+\varepsilon_o)}-\{(\mu+\nu+1)-(l+1)\}\sqrt{(\varepsilon_n-\varepsilon_o)}].\end{aligned}\right\} \tag{69}$$

wobei, wie früher,

$$\gamma = \frac{e^2}{ch_0},$$

ist.

Aus (68) ergibt sich dann die Rekursionsgleichung

$$2[-\gamma Z\,.\,\varepsilon_n+\sqrt{(\varepsilon_n^2-\varepsilon_o^2)}\,.\,(\mu+\nu)]d_\nu = ch_0[-\gamma^2Z^2+\{(\mu+\gamma+1)^2-(l+1)^2\}]d_{\nu+1}. \tag{70}$$

Aus $b_o = 0 = c_o = d_o$, findet man, wie im vorigen Falle

$$-\gamma^2Z^2+(\mu+1)^2-(l+1)^2 = 0,$$

oder

$$\mu = -1+\sqrt{\{(l+1)^2+\gamma^2Z^2\}}, \tag{71}$$

und aus

$$b_{s+1} = 0 = c_{s+1} = d_{s+1},$$

$$-\gamma Z\,.\,\varepsilon_n+\sqrt{(\varepsilon_n^2-\varepsilon_o^2)}\,.\,(\mu+\nu) = 0,$$

oder

$$\varepsilon_n = \varepsilon_o\left\{1-\frac{\gamma^2Z^2}{(\mu+s)^2}\right\}^{-\frac{1}{2}}. \tag{72}$$

(e) *Die unverkurzten Quantenbahnen.*

Fall: (a, α). Nimmt man Bezug auf (A), (45), (43), (41), (36), (27), (25), (17), so findet man

$$\left.\begin{aligned}\psi_1 &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[UP_l^{-m}+VP_{l+1}^{-m}\right]e^{im\varphi},\\ \psi_2 &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[U(l-m)P_l^{-(m+1)}-V(l+m+2)P_{l+1}^{-(m+1)}\right]e^{i(m+1)\varphi},\\ \psi_1' &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[UP_l^{-m}-VP_{l+1}^{-m}\right]e^{im\varphi},\\ \psi_2' &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[U(l-m)P_l^{-(m+1)}+V(l+m+2)P_{l+1}^{-(m+1)}\right]e^{i(m+1)\varphi}.\end{aligned}\right\} \tag{73}$$

Danach berechnet man mit Hilfe von [(39); I] die Bahngleichungen als

$$\left.\begin{aligned}\frac{dt}{d\tau} &= \frac{U^2+V^2}{U^2-V^2},\\ \frac{dz}{cd\tau} &= -\frac{2UV}{U^2-V^2}\cos\theta,\\ \frac{dx}{cd\tau} &= -\frac{2UV}{U^2-V^2}\sin\theta\cos\varphi,\\ \frac{dy}{cd\tau} &= -\frac{2UV}{U^2-V^2}\sin\theta\sin\varphi.\end{aligned}\right\}$$

Mit Bezugnahme auf (A), (46), (44). (42), (36), (27), (25), (40) findet man aber

$$\left.\begin{aligned}\chi_1 &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[UP_l^{-m}-VP_{l+1}^{-m}\right]e^{im\varphi},\\ \chi_2 &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[U(l-m)P_l^{-(m+1)}+V(l+m+2)P_{l+1}^{-(m+1)}\right]e^{i(m+1)\varphi},\\ \chi_1' &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[UP_l^{-m}+VP_{l+1}^{-m}\right]e^{im\varphi},\\ \chi_2' &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[U(l-m)P_l^{-(m+1)}-V(l+m+2)P_{l+1}^{-(m+1)}\right]e^{i(m+1)\varphi}\end{aligned}\right\}\quad(75)$$

Danach ergeben sich nach [(43), I] die Bahngleichungen als

$$\left.\begin{aligned}\frac{dt}{d\tau} &= \frac{U^2+V^2}{U^2-V^2},\\ \frac{dz}{cd\tau} &= +\frac{2UV}{U^2-V^2}\cos\theta,\\ \frac{dx}{cd\tau} &= +\frac{2UV}{U^2-V^2}\sin\theta\cos\varphi,\\ \frac{dy}{cd\tau} &= +\frac{2UV}{U^2-V^2}\sin\theta\sin\varphi.\end{aligned}\right\}\quad(74a)$$

Fall: (b,α). Bezieht man sich hier auf (A), (49), (47), (37), (29), (28), (17) zurück, so hat man

$$\left.\begin{aligned}\psi_1 &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[-U(l-m)P_l^{-(m+1)}-V(l+m+2)P_{l+1}^{-(m+1)}\right]e^{-i(m+1)\varphi},\\ \psi_2 &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[UP_l^{-m}-VP_{l+1}^{-m}\right]e^{-im\varphi},\\ \psi_1' &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[-U(l-m)P_l^{-(m+1)}+V(l+m+2)P_{l+1}^{-(m+1)}\right]e^{-i(m+1)\varphi},\\ \psi_2' &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[UP_l^{-m}+VP_{l+1}^{-m}\right]e^{-im\varphi},\end{aligned}\right\}\quad(76)$$

und die Bahngleichungen nach [(39), I] von genau gleicher Gestalt wie (74).

Mit Rückbezug auf (*A*), (50), (48), (37), (29), (28), (40) findet man aber

$$\left.\begin{aligned}
\chi_1 &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[-U(l-m)P_l^{-(m+1)}+V(l+m+2)P_{l+1}^{-(m+1)}\right]e^{-i(m+1)\varphi},\\
\chi_2 &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[UP_l^{-m}+VP_{l+1}^{-m}\right]e^{-im\varphi},\\
\chi_1' &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[-U(l-m)P_l^{-(m+1)}-V(l+m+2)P_{l+1}^{-(m+1)}\right]e^{-i(m+1)\varphi}\\
\chi_2' &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[UP_l^{-m}-VP_{l+1}^{-m}\right]e^{-im\varphi},
\end{aligned}\right\}\qquad(77)$$

und demnächst die zugehörigen Bahngleichungen nach [(43), I] von derselben Gestalt wie (74*a*).

Fall: (*a*, β). Bezieht man sich auf (*B*), (53), (51), (38), (30) (17), zurück, so findet man

$$\left.\begin{aligned}
\psi_1 &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[vP_{l+1}^{-m}+uP_l^{-m}\right]e^{im\varphi}\\
\psi_2 &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[-v(l+m+2)P_{l+1}^{-(m+1)}+u(l-m)P_l^{-(m+1)}\right]e^{i(m+1)\varphi},\\
\psi_1' &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[vP_{l+1}^{-m}-uP_l^{-m}\right]e^{im\varphi},\\
\psi_2' &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[-v(l+m+2)P_{l+1}^{-(m+1)}-u(l-m)P_l^{-(m+1)}\right]e^{i(m+1)\varphi},
\end{aligned}\right\}\qquad(78)$$

Demnach berechnet man mit Hilfe von [(39) I] die Bahngleichungen als

$$\left.\begin{aligned}
\frac{dt}{d\tau} &= \frac{v^2+u^2}{v^2-u^2}.\\
\frac{dz}{c\,d\tau} &= -\frac{2vu}{v^2-u^2}\cos\theta,\\
\frac{dx}{c\,d\tau} &= -\frac{2vu}{v^2-u^2}\sin\theta\cos\varphi,\\
\frac{dy}{c\,d\tau} &= -\frac{2vu}{v^2-u^2}\sin\theta\sin\varphi,
\end{aligned}\right\}\qquad(79)$$

Mit Bezugnahme auf (B), (54), (52), (38), (30), (40), findet man aber

$$\left.\begin{aligned}
\lambda_1 &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\ \frac{1}{r}\left[vP_{l+1}^{-m}-uP_l^{-m}\right]e^{im\varphi},\\
\chi_2 &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\ \frac{1}{r}\left[-v(l+m+2)P_{l+1}^{-(m+1)}-u(l-m\quad {}_l^{-(m+1)}\right]e^{i(m+1)\varphi},\\
\chi_1' &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\ \frac{1}{r}\left[vP_{l+1}^{-m}+uP_l^{-m}\right]e^{im\varphi},\\
\chi_2' &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\ \frac{1}{r}\left[-v(l+m+2)P_{l+1}^{-(m+1)}+u(l-m)P_l^{-(m+1)}\right]e^{i(m+1)\varphi},
\end{aligned}\right\}\qquad(80)$$

Demzufolge ergeben sich nach [(43), I] die Bahngleichungen als

$$\left.\begin{aligned}
\frac{dt}{d\tau} &= \frac{v^2+u^2}{v^2-u^2},\\
\frac{dz}{cd\tau} &= +\frac{2vu}{v^2-u^2}\cos\theta,\\
\frac{dx}{cd\tau} &= +\frac{2vu}{v^2-u^2}\sin\theta\cos\varphi,\\
\frac{dy}{cd\tau} &= +\frac{2vu}{v^2-u^2}\sin\theta\sin\varphi,
\end{aligned}\right\}\qquad(79a)$$

Fall: (b, β). Eine Bezugnahme auf (B), (57), (55), (39), (31), (17), liefert für diesen Fall zunächst

$$\left.\begin{aligned}
\psi_1 &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\ \frac{1}{r}\left[v(l+m+2)P_{l+1}^{-(m+1)}+u(l-m)P_l^{-(m+1)}\right]e^{-i(m+1)\varphi},\\
\psi_2 &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\ \frac{1}{r}\left[vP_{l+1}^{-m}-uP_l^{-m}\right]e^{-im\varphi}.\\
\psi_1' &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\ \frac{1}{r}\left[v(l+m+2)P_{l+1}^{-(m+1)}-u(l-m)P_l^{-(m+1)}\right]e^{-i(m+1)\varphi},\\
\psi_2' &= \exp\left(-\frac{\varepsilon_n}{h_0}t\right)\cdot\ \frac{1}{r}\left[vP_{l+1}^{-m}+uP_l^{-m}\right]e^{-im\varphi},
\end{aligned}\right\}\qquad(81)$$

und dann nach [(39), I] die Bahngleichungen von genau gleicher Gestalt wie (79).

Mit Rückbezug auf (*B*), (58), (56), (39), (31, (40), findet man aber

$$\left.\begin{aligned}
\chi_1 &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[v(l+m+2)P_{l+1}^{-(m+1)}-u(l-m)P_l^{-(m+1)}\right]e^{-i(m+1)\varphi},\\
\chi_2 &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[vP_{l+1}^{-m}+uP_l^{-m}\right]e^{-i(m+1)\varphi},\\
\chi_1' &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[v(l+m+2)P_{l+1}^{-(m+1)}+u(l-m)P_l^{-(m+1)}\right]e^{-i(m+1)\varphi},\\
\chi_2' &= \exp\left(+\frac{\varepsilon_n}{h_0}t\right)\cdot\frac{1}{r}\left[vP_{l+1}^{-m}-uP_l^{-m}\right]e^{-im\varphi},
\end{aligned}\right\}\qquad(82)$$

und die zugehörigen Bahngleichungen nach [(43), I] von derselben Gestalt, wie (79*a*).

(*f*) *Die entsprechenden klassischen Bahnen.*

Wir beziehen uns auf [(*B*), I], setzen dort

$$\varphi = \frac{Ze}{r} = V,\ A_x = 0 = A_y = A_z, \qquad (83)$$

und so bekommen

$$\frac{\varepsilon_0^2}{c^2} = \left(\frac{\partial S}{c\partial t}+\frac{eV}{c}\right)^2-\sum_x\left(\frac{\partial S}{\partial x}\right)^2=\left(\frac{\partial S}{c\partial t}+\frac{eV}{c}\right)^2-\left\{\left(\frac{\partial S}{\partial r}\right)^2+\left(\frac{\partial S}{r\partial\theta}\right)^2+\left(\frac{\partial S}{r\sin\theta\partial\varphi}\right)^2\right\}, \qquad (84)$$

Mit her Setzung

$$S = -\varepsilon t + w(r), \qquad (85)$$

hat man daraus

$$c^2S'^2=\left(-\varepsilon+\frac{Ze^2}{r}\right)^2-\varepsilon_0^2, \qquad (86)$$

und nach [(*A*), I] die Bahngleichungen als

$$\left.\begin{aligned}
\varepsilon_0\frac{dt}{d\tau} &= \frac{\partial S}{\partial t}+eV=\left(-\varepsilon+\frac{Ze^2}{r}\right),\\
-\varepsilon_0\frac{dz}{cd\tau} &= c\frac{\partial S}{\partial z}=c\frac{\partial S}{\partial r}\frac{\partial r}{\partial z}=cS'\cos\theta\\
-\varepsilon_0\frac{dx}{cd\tau} &= c\frac{\partial S}{\partial x}=cS'\sin\theta\cos\varphi,\\
-\varepsilon_0\frac{dy}{cd\tau} &= c\frac{\partial S}{\partial y}=cS'\sin\theta\sin\varphi,
\end{aligned}\right\}\qquad(87)$$

Mit gleichzeitigem Umschlagen des Vorzeichens von e bzw. ε_0 hat man aus (84)

$$\frac{\varepsilon_0^2}{c^2}=\left(\frac{\partial S}{c\partial t}-\frac{eV}{c}\right)^2-\sum_x\left(\frac{\partial S}{\partial x}\right)^2, \qquad (88)$$

und nach $[(A), \mathrm{I}]$ die Bahngleichungen als

$$\left.\begin{aligned} -\varepsilon_0 \frac{dt}{d\tau} &= \frac{\partial S}{\partial t} - eV = \varepsilon - \frac{Ze^2}{r} \\ \varepsilon_0 \frac{dz}{cd\tau} &= c\frac{\partial S}{\partial z} = cS' \cos\theta \\ \varepsilon_0 \frac{dx}{cd\tau} &= c\frac{\partial S}{\partial x} = cS' \sin\theta \cos\varphi, \\ \varepsilon_0 \frac{dy}{cd\tau} &= c\frac{\partial S}{\partial y} = cS' \sin\theta \sin\varphi, \end{aligned}\right\} \tag{89}$$

wobei

$$S = \varepsilon t - w(r), \tag{90}$$

zu setzen ist.

Setzt man in (87) gemäss (85) u (86)

$$S' = +\sqrt{\left\{\left(-\varepsilon_n + \frac{Ze^2}{r}\right)^2 - \varepsilon_0^2\right\}} = +w', \tag{91}$$

aber in (89) gemäss (90) u (86)

$$-S' = +\sqrt{\left\{\left(+\varepsilon_n - \frac{Ze^2}{r}\right)^2 - \varepsilon_0^2\right\}} = +w', \tag{92}$$

so ergibt sich die gleiche Gestalt für die klassischen Bahngleichungen (87) bzw. (89), nämlich,

$$\left.\begin{aligned} &\varepsilon_0 \frac{dt}{cd\tau} = -\varepsilon + \frac{Ze^2}{r}, \\ &\varepsilon_0\left(\frac{dz}{cd\tau}, \frac{dx}{cd\tau}, \frac{dy}{cd\tau}\right) = -cw' (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi) \end{aligned}\right\} \tag{93}$$

(*g*) *Die Quantenbahnen bis auf die erste Näherung.*

(*i*) *Reduktion von* (*A*) *bis auf die erste Näherung.*

Leitet man aus (A) Gleichungen zweiter Ordnung in U bzw. V her, so lauten sie

$$\left.\begin{aligned} c^2h_0^2\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right]U &= \left[\left(-\varepsilon_n + \frac{Ze^2}{r}\right)^2 - \varepsilon_0^2\right]U \\ &\quad - \frac{Ze^2}{r^2[(-\varepsilon_n + Ze^2/r) + \varepsilon_0]} \cdot c^2h_0^2\left[\frac{d}{dr} - \frac{l+1}{r}\right]U, \\ c^2h_0^2\left[\frac{d^2}{dr^2} - \frac{(l+1)(l+2)}{r^2}\right]V &= \left[-\varepsilon_n + \frac{Ze^2}{r}\right)^2 - \varepsilon_0^2\Big]V \\ &\quad - \frac{Ze^2}{r^2[(-\varepsilon_n + Ze^2/r) - \varepsilon_0]} \cdot c^2h_0^2\left[\frac{d}{dr} + \frac{l+1}{r}\right]V. \end{aligned}\right\} \tag{94}$$

Wir schreiben sie dann, wie es folgt, um

$$\left.\begin{aligned} c^2h_0^2\left[\left(\frac{U'}{U}\right)' + \left(\frac{U'}{U}\right)^2 - \frac{l(l+1)}{r^2}\right] &= \left[\left(-\varepsilon_n + \frac{Ze^2}{r}\right)^2 - \varepsilon_0^2\right] \\ &\quad - \frac{Ze^2}{r^2[(-\varepsilon_n + Ze^2/r) + \varepsilon_0]} \cdot c^2h_0^2\left[\frac{U'}{U} - \frac{l+1}{r}\right], \\ c^2h_0^2\left[\left(\frac{V'}{V}\right)' + \left(\frac{V'}{V}\right)^2 - \frac{(l+1)(l+2)}{r^2}\right] &= \left[\left(-\varepsilon_n + \frac{Ze^2}{r}\right)^2 - \varepsilon_0^2\right] \\ &\quad - \frac{Ze^2}{r^2[(-\varepsilon_n + Ze^2/r) - \varepsilon_0]} \cdot c^2h_0^2\left[\frac{V'}{V} + \frac{l+1}{r}\right]. \end{aligned}\right\} \tag{95}$$

Dabei bedeutet der Strich eine Differentiation nach r.

Zwecks Näherung machen wir nun in (95) die Ansätze

$$\left.\begin{aligned} \lim_{h_0 \to 0}(h_0 \ln U)' &= S' + h_0\alpha_1' + h_0^2\alpha_2' + \dots \dots \\ \lim_{h_0 \to 0}(h_0 \ln V)' &= S' + h_0\beta_1' + h_0^2\beta_2' + \dots \dots \\ \lim_{h_0 \to 0}[h_0(l+1)] &= a \to 0. \end{aligned}\right\} \tag{96}$$

Man erhält dann

$$\left.\begin{aligned} & c^2[h_0(S'' + h_0\alpha_1'' + h_0^2\alpha_2'' + \dots) + (S' + h_0\alpha_1' + h_0^2\alpha_2' + \dots)^2 - 0] \\ & = \left[\left(-\varepsilon_n + \frac{Ze^2}{r}\right)^2 - \varepsilon_0^2\right] - \frac{Ze^2}{r^2[(-\varepsilon_n + Ze^2/r) + \varepsilon_0]} c^2h_0[(S' + h_0\alpha_1' + h_0^2\alpha_2' + \dots) - 0], \\ & c^2[h_0(S'' + h_0\beta_1'' + h_0^2\beta_2'' + \dots) + (S' + h_0\beta_1' + h_0^2\beta_2' + \dots)^2 - 0] \\ & = \left[\left(-\varepsilon_n + \frac{Ze^2}{r}\right)^2 - \varepsilon_0^2\right] - \frac{Ze^2}{r^2[(-\varepsilon_n + Ze^2/r) - \varepsilon_0]} \cdot c^2h_0[(S' + h_0\beta_1' + h_0^2\beta_2' + \dots) + 0]. \end{aligned}\right\} \tag{97}$$

Daher entnimmt man bei nullter Näherung

$$c^2S'^2 = \left[\left(-\varepsilon_n + \frac{Ze^2}{r}\right)^2 - \varepsilon_0^2\right], \tag{98}$$

bei erster Näherung aber

$$\left.\begin{aligned} \alpha_1' &= -\tfrac{1}{2}\left[\frac{S''}{S'} + \frac{Ze^2}{r^2[(-\varepsilon_n + Ze^2/r) + \varepsilon_0]}\right], \\ \beta_1' &= -\tfrac{1}{2}\left[\frac{S''}{S'} + \frac{Zc^2}{r^2[(-\varepsilon_n + Ze^3/r) - \varepsilon_0]}\right]. \end{aligned}\right\} \tag{99}$$

Greift man nun wieder zu (A) zurück, so erhält man im Limes $(h_o \to 0)$ mittels (96)

$$\left.\begin{aligned}\left\{\left(-\varepsilon_n+\frac{Ze^2}{r}\right)-\varepsilon_o\right\}\left(\frac{U}{V}\right)_L&=\left[ch_o\left\{\frac{V'}{V}+\frac{l+1}{r}\right\}\right]_L=c\{S'+h_o\beta_1'+h_o^2\beta_2'+\dots)+0\},\\ \left\{\left(-\varepsilon_n+\frac{Ze^2}{r}\right)+\varepsilon_o\right]\left(\frac{V}{U}\right)_L&=\left[ch_o\left\{\frac{U'}{U}-\frac{l+1}{r}\right\}\right]_L=c\{S'+h_o\alpha_1'+h_o^2\alpha_2'+\dots)-0\}.\end{aligned}\right\}\quad(100)$$

Bis auf die erste Näherung hat man daher

$$\left.\begin{aligned}\left(-\varepsilon_n+\frac{Ze^2}{r}\right)\left(\frac{U}{V}+\frac{V}{U}\right)_L-\varepsilon_o\left(\frac{U}{V}-\frac{V}{U}\right)_L&=2cS'+ch_o(\beta_1'+\alpha_1'),\\ \left(-\varepsilon_n+\frac{Ze^2}{r}\right)\left(\frac{U}{V}-\frac{V}{U}\right)_L-\varepsilon_o\left(\frac{U}{V}+\frac{V}{U}\right)_L&=ch_o(\beta_1'-\alpha_1').\end{aligned}\right\}\quad(101)$$

Aus (99) folgt :

$$\beta_1'-\alpha_1'=-\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_o}{c^2S'^2}\quad(102)$$

und mit Hilfe von (98) noch

$$\beta_1'+\alpha_1'=0,\quad(103)$$

Demnächst hat man

$$\left.\begin{aligned}\left(\frac{U}{V}+\frac{V}{U}\right)_L&=\frac{2}{c^2S'^2}\left[\left(-\varepsilon_n+\frac{Ze^2}{r}\right)cS'-\frac{ch_o}{2}\,\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0^2}{c^2S'^2}\right],\\ \left(\frac{U}{V}-\frac{V}{U}\right)_L&=\frac{2}{c^2S'^2}\left[\varepsilon_o cS'-\frac{ch_o}{2}\,\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_o(-\varepsilon_n+Ze^2/r)}{c^2S'^2}\right].\end{aligned}\right\}\quad(104)$$

(ii) *Reduktion von (B) bis auf die erste Näherung.*

Folgert man, wie früher, die Gleichungen zweiter Ordnung aus (B), schreibt sie als

$$\left.\begin{aligned}c^2h_o^2\left[\left(\frac{u'}{u}\right)'+\left(\frac{u'}{u}\right)^2-\frac{l(l+1)}{r^2}\right]&=\left[\left(-\varepsilon_n+\frac{Ze^2}{r}\right)^2-\varepsilon_o^2\right]\\ &\quad-\frac{Ze^2}{r^2[(-\varepsilon_n+Ze^2/r)-\varepsilon_o]}c^2h_o^2\left[\frac{u'}{u}-\frac{l+1}{r}\right],\\ c^2h_o^2\left[\left(\frac{v'}{v}\right)'+\left(\frac{v'}{v}\right)^2-\frac{(l+1)\,(l+2)}{r^2}\right]&=\left[\left(-\varepsilon_n+\frac{Ze^2}{r}\right)^2-\varepsilon_o^2\right]\\ &\quad-\frac{Ze^2}{r^2[(-\varepsilon_n+Ze^2/r)-\varepsilon_o]}c^2h_o^2\left[\frac{v'}{v}+\frac{l+1}{r}\right].\end{aligned}\right\}\quad(105)$$

und zwecks Näherung darin setzt

$$\left.\begin{aligned}&\lim_{h_o\to 0}(h_o\ln u)'=S'+h_o\gamma_1'+h_o^2\gamma_2'+\dots,\\ &\lim_{h_o\to 0}(h_o\ln v)'=S'+h_o\delta_1'+h_o^2\delta_2'+\dots.\\ &\lim_{h_o\to 0}[h_o(l+1)]=a\to 0,\end{aligned}\right\}\quad(106)$$

so erhält man

$$\left.\begin{aligned}
&c^2[h_o(S'' + h_o\gamma''_1 + h_o^2\gamma''_2 + \ldots) + (S' + h_o\gamma'_1 + h_o^2\gamma'_2 + \ldots)^2]\\
&\quad = \left[\left(-\varepsilon_n + \frac{Ze^2}{r}\right)^2 - \varepsilon_o^2\right] - \frac{Ze^2}{r^2[(-\varepsilon_n + Ze^2/r) + \varepsilon_o]} \cdot c^2 h_o[S + h_o\gamma'_1 + h_o^2\gamma_2 + \ldots]\\
&c^2[h_o(S' + h_o\delta''_1 + h_o^2\delta''_2 + \ldots) + (S + h_o\delta'_1 + h_o^2\delta'_2 + \ldots)^2]\\
&\quad = \left[\left(-\varepsilon_n + \frac{Ze^2}{r}\right)^2 + \varepsilon_o^2\right] - \frac{Ze^2}{r^2[(-\varepsilon_n + Ze^2/r) + \varepsilon_o]} \cdot c^2 h_o[S' + h_o\delta'_1 + h_o^2\delta'_2 + \ldots].
\end{aligned}\right\} \tag{107}$$

Daher findet man bei nullter Naherung dieselbe Bedingung (98) oben, bei erster Näherung aber

$$\left.\begin{aligned}
\gamma'_1 &= -\tfrac{1}{2}\left[\frac{S''}{S'} + \frac{Ze^2}{r^2\{(-\varepsilon_n + Ze^2/r) - \varepsilon_o\}}\right] = \beta'_1\\
\delta'_1 &= -\tfrac{1}{2}\left[\frac{S''}{S'} + \frac{Ze^2}{r^2\{(-\varepsilon_n + Ze^2/r) + \varepsilon_o\}}\right] = \alpha'_1
\end{aligned}\right\} \tag{108}$$

Aus (B) erhält man im Limes $(h_o \to 0)$ mittels (106)

$$\left.\begin{aligned}
\left\{\left(-\varepsilon_n + \frac{Ze^2}{r}\right) - \varepsilon_o\right\}\left(\frac{v}{u}\right)_L &= c\{(S' + h_o\gamma'_1 + h_o^2\gamma'_2 + \ldots) - 0\},\\
\left\{\left(-\varepsilon_n + \frac{Ze^2}{r}\right) + \varepsilon_o\right\}\left(\frac{u}{v}\right)_L &= c\{(S' + h_o\delta'_1 + h_o^2\delta'_2 + \ldots) + 0\}.
\end{aligned}\right\} \tag{109}$$

Bis auf die erste Näherung hat man daher

$$\left.\begin{aligned}
\left(-\varepsilon_n + \frac{Ze^2}{r}\right)\left(\frac{v}{u} + \frac{u}{v}\right)_L - \varepsilon_o\left(\frac{v}{u} - \frac{u}{v}\right)_L &= 2cS' + ch_o(\gamma'_1 + \delta'_1),\\
\left(-\varepsilon_n + \frac{Ze^2}{r}\right)\left(\frac{v}{u} - \frac{u}{v}\right)_L - \varepsilon_o\left(\frac{v}{u} + \frac{u}{v}\right)_L &= ch_o(\gamma'_1 - \delta'_1).
\end{aligned}\right\} \tag{110}$$

Nach (108), (102), (103) gilt aber

$$\left.\begin{aligned}
\gamma'_1 + \delta'_1 &= 0,\\
\gamma'_1 - \delta'_1 &= -\frac{Ze^2}{r^2} \cdot \frac{\varepsilon_o}{c^2 S'^2}.
\end{aligned}\right\} \tag{111}$$

Demnach ermittelt man

$$\left.\begin{aligned}
\left(\frac{v}{u} + \frac{u}{v}\right)_L &= \frac{2}{cS'^2}\left[\left(-\varepsilon_n + \frac{Ze^2}{r}\right)S' - \frac{h_o}{2} \cdot \frac{Ze^2}{r^2} \cdot \frac{\varepsilon_o}{c^2 S'^2}\right],\\
\left(\frac{v}{u} - \frac{u}{v}\right)_L &= \frac{2}{cS'^2}\left[\varepsilon_o \cdot S' - \frac{h_o}{2} \cdot \frac{Ze^2}{r^2} \cdot \frac{\varepsilon_o(-\varepsilon_n + Ze^2/r)}{c^2 S'^2}\right].
\end{aligned}\right\} \tag{112}$$

(iii) Die Bahngleichungen bis auf die erste Näherung.

Die Fälle: (a, α) *bzw.* (b, α).

Ist der Zeitfaktor $\exp\left(-\frac{\varepsilon_n}{h_0}t\right)=\underset{-}{Z}$, so sind die unverkürzten Bahngleichungen die durch (74) gegebenen. Bis auf die erste Näherung haben sie also nach (104) die Gestalt

$$\left.\begin{aligned}
\frac{dt}{d\tau}&=\frac{\left(\frac{U}{V}+\frac{V}{U}\right)_L}{\left(\frac{U}{V}-\frac{V}{U}\right)_L}=\frac{\left(-\varepsilon_n+\frac{Ze^2}{r}\right)w'-\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon^2}{c^2w'^2}}{\varepsilon_0 w'-\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0(-\varepsilon_n+Ze^2/r)}{e^2w'^2}},\\
\left(\frac{dz}{cd\tau},\frac{dx}{cd\tau},\frac{dy}{cd\tau}\right)&=-(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi\ \frac{2}{\left(\frac{U}{V}-\frac{V}{U}\right)_L}\\
&=-(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)\cdot\frac{cw'^2}{\varepsilon_0 w'-\frac{h_0}{2}\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0(-\varepsilon_n+Ze^2/r)}{c^2w'^2}},
\end{aligned}\right\}\quad(113)$$

indem gemäss (91) $S'=+w'$ zu setzen ist.

Ist aber der Zeitfaktor $\exp\left(+\frac{\varepsilon_n}{h_0}t\right)=\underset{+}{Z}$, so sind die unverkürzten Bahngleichungen die durch (74*a*) gegebenen. Bis auf die erste Näherung haben sie nach (104) also die Gestalt

$$\left.\begin{aligned}
\frac{dt}{d\tau}&=\frac{\left(\frac{U}{V}+\frac{V}{U}\right)_L}{\left(\frac{U}{V}-\frac{V}{U}\right)_L}=\frac{\left(-\varepsilon_n+\frac{Ze^2}{r}\right)w'+\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0^2}{c^2w'^2}}{\varepsilon_0 w'+\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0(-c_n+Ze^2/r)}{c^2w'^2}},\\
\left(\frac{dz}{cd\tau},\frac{dx}{cd\tau},\frac{dy}{cd\tau}\right)&=+(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)\cdot\frac{2}{\left(\frac{U}{V}-\frac{V}{U}\right)_L}\\
&=-(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)\frac{cw'^2}{\varepsilon_0 w'+\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0(-\varepsilon_n+Ze^2/r)}{c^2w'^2}}
\end{aligned}\right\}\quad(114)$$

weil gemäss (92) $S'=-w'$ gesetzt wird.

In nullter Näherung stimmen also (113) und (114) miteinander und mit (93) überein. In erster Näherung weichen sie aber voneinander ab.

Die Fälle: (a, β) *bzw.* (b, β).

Im Falle des Zeitfaktors: $\exp\left(-\frac{\varepsilon_n}{h_0}t\right)=\underset{-}{Z}$ sind die unverkürzten Bahngleichungen die durch (79) gegebenen. Bis auf die erste Naherung haben sie nach (112) also die Gestalt

$$\left.\begin{aligned}
\frac{dt}{d\tau}&=\frac{\left(\frac{v}{u}+\frac{u}{v}\right)_L}{\left(\frac{v}{u}-\frac{u}{v}\right)_L}=\frac{\left(-\varepsilon_n+\frac{Ze^2}{r}\right)w'-\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0^2}{c^2w'^2}}{\varepsilon_0 w'-\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0(-\varepsilon_n+Ze^2/r)}{c^2w'^2}},\\
\left(\frac{dz}{cd\tau},\frac{dx}{cd\tau},\frac{dy}{cd\tau}\right)&=-(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)\cdot\frac{2}{\left(\frac{v}{u}-\frac{u}{v}\right)_L}\\
&=-(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)\cdot\frac{cw'^2}{\varepsilon_0 w'-\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0(-\varepsilon_n+Ze^2/r),}{c^2w'^2}}
\end{aligned}\right\}\quad(115)$$

wobei gemäss (91) $S'=+w'$ gesetzt worden ist.

Im Falle aber des Zeitfaktors : $\exp\left(+\frac{\varepsilon_n}{h_0}t\right)=Z_+$ sind die unverkürzten Bahngleichungen die durch (79*a*) gegebenen. Bis auf die erst Naberung haben sie nach (112) die Gestalt

$$\left.\begin{aligned}
\frac{dt}{d\tau}&=\frac{\left(\frac{v}{u}+\frac{u}{v}\right)_L}{\left(\frac{v}{u}-\frac{u}{v}\right)_L}=\frac{\left(-\varepsilon_n+\frac{Ze^2}{r}\right)w'+\frac{h_0}{2}\cdot\frac{Ze^3}{r^2}\cdot\frac{\varepsilon_0^2}{c^2w'^2}}{\varepsilon_0 w'+\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0(-\varepsilon_n+Ze^2/r)}{c^2w'^2}},\\
\left(\frac{dz}{cd\tau},\frac{dx}{cd\tau},\frac{dy}{cd\tau}\right)&=+(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)\cdot\frac{2}{\left(\frac{v}{u}-\frac{u}{v}\right)_L}\\
&=-(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)\ \frac{cw'^2}{\varepsilon_0 w'+\frac{h_0}{2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\varepsilon_0(-\varepsilon_n+Ze^2/r),}{c^2w'^2}}
\end{aligned}\right\}\quad(116)$$

wobei nach (92) $S'=-w'$ gesetzt wird.

In nullter Näherung stimmen also (115) und (116) miteinander und mit (93) überein. In erster Näherung weichen sie aber voneinander ab.

Das Elektron im konstanten elektrischen Felde nach der Fassung (*i*) $[B_2, A_2]$ *bzw.* (*ii*) $[B_2(a), A_2(a)]$ *des Teils I.*

(*i*) *Fassung* $[B_2, A_2]$.

Mit dem Ansatz:

$$\varphi=\varphi_0-Ez=V,\ A_x=0=A_y=A_z,\tag{117}$$

hat man nach [(35), I]

$$\left.\begin{aligned}\left[\left(h_0\frac{\partial}{c\partial t}+\frac{eV}{c}\right)^2-h_0^2\nabla^2+\frac{ch_0}{c}\sigma_z E\right]\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}&=\frac{\varepsilon_0{}^2}{c^2}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix},\\ \left[\left(h_0\frac{\partial}{c\partial t}+\frac{eV}{c}\right)+h_0(\vec{\sigma}\vec{\nabla})\right]\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}&=+\frac{\varepsilon_0}{c}\begin{vmatrix}\psi_1'\\ \psi_2'\end{vmatrix}.\end{aligned}\right\}\qquad(118)$$

Mit den Setzung

$$(\psi_1,\psi_2;\psi_1',\psi_2')=(f,g;f',g')\exp\frac{-\varepsilon t+xp_1+yp_2}{h_0},\qquad(119)$$

ergeben sich daraus

$$\left.\begin{aligned}\left[\frac{d^2}{dz^2}+\left\{\frac{\varepsilon_0{}^2+c^2K^2}{c^2h_0{}^2}-\frac{eE}{ch_0}\right\}-\left(\frac{eE}{ch_0}\right)^2\left(z+\frac{\varepsilon'}{eE}\right)^2\right]f&=0,\\ \left[\frac{d^2}{dz^2}+\left\{\frac{\varepsilon_0{}^2+c^2K^2}{c^2h_0{}^2}+\frac{eE}{ch_0}\right\}-\left(\frac{eE}{ch_0}\right)^2\left(z+\frac{\varepsilon'}{eE}\right)^2\right]g&=0,\end{aligned}\right\}\qquad(120)$$

$$\left.\begin{aligned}\frac{c}{\varepsilon_0}\left[-\frac{eE}{c}\left(z+\frac{\varepsilon'}{eE}\right)f+h_0\frac{df}{dz}+(p_1-ip_2)g\right]&=f',\\ \frac{c}{\varepsilon_0}\left[-\frac{eE}{c}\left(z+\frac{\varepsilon'}{eE}\right)g-h_0\frac{dg}{dz}+(p_1+ip_2)f\right]&=g',\end{aligned}\right\}\qquad(121)$$

wobei zur Abkürzung gesetzt werden

$$\left.\begin{aligned}\varepsilon-e\varphi_0&=\varepsilon',\\ p_1^2+p_2^2&=K^2,\end{aligned}\right\}\qquad(122)$$

Mit der Setzung

$$(2n)\cdot\frac{eE}{ch_0}=\frac{\varepsilon_0{}^2+c^2K^2}{c^2h_0{}^2},\qquad(123)$$

sieht man an (120)

$$\left.\begin{aligned}f&=a\varphi_{n-1}(\xi),\quad g=b\varphi_n(\xi),\\ \xi&=\left(\frac{eE}{ch_0}\right)^{\frac{1}{2}}\cdot\left(z+\frac{\varepsilon'}{eE}\right),\end{aligned}\right\}\qquad(124)$$

wo φ_{n-1}, φ_n die bekannten Hermite'schen Funktionen sind. Mit Hilfe bekannter Eigenschaften von Hermite'schen Funktionen folgert man nun aus (121) leicht

$$f'=a'\varphi_n(\xi),\quad g'=b'\varphi_{n-1}(\xi),\qquad(125)$$

wo

$$\left.\begin{aligned}a'&=\frac{c}{\varepsilon_0}\left\{-a\frac{\sqrt{(\varepsilon_0{}^2+e^2K^2)}}{c}+b(p_1-ip_2)\right\},\\ b'&=\frac{c}{\varepsilon_0}\left\{-b\frac{\sqrt{(\varepsilon_0{}^2+c^2K^2)}}{c}+a(p_1+ip_2)\right\},\end{aligned}\right\}\qquad(126)$$

Berechnet man nun die Bahngleichungen nach [(89). I], so findet man für sie

$$\left.\begin{aligned} \frac{dt}{d\tau} &= -\frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{2\varepsilon_0}\left(\frac{\varphi_{n-1}}{\varphi_n} + \frac{\varphi_n}{\varphi_{n-1}}\right), \\ -\frac{dz}{cd\tau} &= \frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{2\varepsilon_0}\left(\frac{\varphi_{n-1}}{\varphi_n} - \frac{\varphi_n}{\varepsilon_{n-1}}\right), \\ -\frac{dx}{cd\tau} &= \frac{c}{\varepsilon_0}p_1, \quad -\frac{dy}{cd\tau} = \frac{c}{\varepsilon_0}p_2. \end{aligned}\right\} \tag{127}$$

Die klassische Bahn.

Wir beziehen uns auf [(*B*), I], machen dort den Ansatz (117) und so bekommen

$$\frac{\varepsilon_0^2}{c^2} = \left(\frac{dS}{c\partial t} + \frac{eV}{c}\right)^2 - \sum_x \left(\frac{\partial S}{\partial x}\right)^2. \tag{128}$$

Mit der Setzung:

$$S = -\varepsilon t + xp_1 + yp_2 + p_3(z), \tag{129}$$

hat man daraus

$$\left(\frac{\partial S}{\partial z}\right)^2 = \left(\frac{eE}{c}\right)^2\left(z + \frac{\varepsilon'}{eE}\right)^2 - \frac{\varepsilon_0^2 + c^2K^2}{c^2} = (S')^2.$$

Nach [(*A*), I] lauten die klassischen Gleichungen dann

$$\left.\begin{aligned} +\varepsilon_0\frac{dt}{d\tau} &= \frac{\partial S}{\partial t} + eV = -(eE)\left(z + \frac{\varepsilon'}{eE}\right), \\ -\frac{\varepsilon_0}{c^2}\frac{dx}{d\tau} &= \frac{\partial S}{\partial x} = p_1; \quad -\frac{\varepsilon_0}{c^2}\frac{dy}{d\tau} = \frac{\partial S}{\partial y} = p_2, \\ -\frac{\varepsilon_0}{c^2}\frac{dz}{d\tau} &= \frac{\partial S}{\partial z} = +\sqrt{\left\{\left(\frac{eE}{c}\right)^2\left(z + \frac{\varepsilon'}{eE}\right)^2 - \frac{\varepsilon_0^2 + c^2K^2}{c^4}\right\}} = \frac{dp_3}{dz} \end{aligned}\right\} \tag{130}$$

Reduktion von (120) bis auf die erste Näherung.

Schreibt man (120) wie

$$\left.\begin{aligned} h_0^2\left[\left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2\right] + \frac{\varepsilon_0^2 + c^2K^2}{c^2} - h_0\frac{eE}{c} - \left(\frac{eE}{c}\right)^2\left(z + \frac{\varepsilon'}{eE}\right)^2 &= 0, \\ h_0^2\left[\left(\frac{g'}{g}\right)' + \left(\frac{g'}{g}\right)^2\right] + \frac{\varepsilon_0^2 + c^2K^2}{c^2} + h_0\frac{eE}{c} - \left(\frac{eE}{c}\right)^2\left(z + \frac{\varepsilon'}{eE}\right)^2 &= 0. \end{aligned}\right\} \tag{131}$$

und setzt dort zwecks Näherung

$$\left.\begin{aligned} \operatorname*{Limes}_{h_0 \to 0}\,(h_0 \ln f)' &= S' + h_0\alpha_1' + h_0^2\alpha_2' + \dots \quad, \\ \text{Limes}\,(h_0 \ln g)' &= S' + h_0\beta_1' + h_0^2\beta_2' + \dots \quad, \end{aligned}\right\} \tag{102}$$

so erhält man

$$\left.\begin{aligned} &h_o(S'' + h_o\alpha_1'' + h_o^2\alpha_2'' + \ldots) + (S' + h_o\alpha_1' + h_o^2\alpha_2' + \ldots)^2 + \frac{\varepsilon_0^2 + c^2K^2}{c^2} \\ &\qquad\qquad - h_o\frac{eE}{c} - \left(\frac{eE}{c}\right)^2\left(z + \frac{\varepsilon'}{eE}\right)^2 = 0, \\ &h_o(S'' + h_o\beta_1'' + h_o^2\beta_2'' + \ldots) + (S' + h_o\beta_1' + h_o^2\beta_2' + \ldots)^2 + \frac{\varepsilon_0^2 + c^2K^2}{c^2} \\ &\qquad\qquad + h_o\frac{eE}{c} - \left(\frac{eE}{c}\right)^2\left(z + \frac{\varepsilon'}{eE}\right)^2 = 0. \end{aligned}\right\} \tag{133}$$

Daher entnimmt man in nullter Näherung

$$S'^2 + \frac{\varepsilon_0^2 + c^2K^2}{c^2} - \left(\frac{eE}{c}\right)\left(z + \frac{\varepsilon'}{eE}\right)^2 = 0, \tag{134}$$

in erster Näherung aber

$$\left.\begin{aligned} \alpha_1' &= -\tfrac{1}{2}\left(\frac{S''}{S'} - \frac{eE}{c}\cdot\frac{1}{S'}\right), \\ \beta_1' &= -\tfrac{1}{2}\left(\frac{S''}{S'} + \frac{eE}{c}\cdot\frac{1}{S'}\right). \end{aligned}\right\} \tag{135}$$

Reduktion von (121). Aus (121) hat man mit Rücksicht auf (124), (125), (126).

$$\left.\begin{aligned} h_o\frac{d}{dz}(\ln\varphi_{n-1}) - \frac{eE}{c}\left(z + \frac{\varepsilon'}{eE}\right) &= -\frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{c}\cdot\frac{\varphi_n}{\varphi_{n-1}}, \\ h_o\frac{d}{dz}(\ln\varphi_n) + \frac{eE}{c}\left(z + \frac{\varepsilon'}{eE}\right) &= +\frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{c}\cdot\frac{\varphi_{n-1}}{\varphi_n}. \end{aligned}\right\} \tag{136}$$

Daher hat man in Limes

$$\left.\begin{aligned} -\frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{c}\left(\frac{\varphi_n}{\varphi_{n-1}}\right)_L &= (S' + h_o\alpha_1' + h_o^2\alpha_2' + \ldots) - \frac{eE}{c}\left(z + \frac{\varepsilon'}{eE}\right), \\ +\frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{c}\left(\frac{\varphi_{n-1}}{\varphi_n}\right)_L &= (S' + h_o\beta_1' + h_o^2\beta_2' + \ldots) + \frac{eE}{c}\left(z + \frac{\varepsilon'}{eE}\right), \end{aligned}\right\} \tag{137}$$

und bis auf die erste Näherung

$$\left.\begin{aligned} \frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{c}\left(\frac{\varphi_{n-1}}{\varphi_n} + \frac{\varphi_n}{\varphi_{n-1}}\right)_L &= 2\Big[\frac{h_o}{2}(\beta_1' - \alpha_1') + \frac{eE}{e}\left(z + \frac{\varepsilon'}{eE}\right) \\ &= 2\left[-\frac{h_o}{2}\cdot\frac{eE}{c}\cdot\frac{1}{S'} + \frac{eE}{c}\left(z + \frac{\varepsilon'}{eE}\right)\right], \\ \frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{c}\left(\frac{\varphi_{n-1}}{\varphi_n} - \frac{\varphi_n}{\varphi_{n-1}}\right)_L &= 2\left[S' + \frac{h_o}{2}(\alpha_1' + \beta_1')\right] \\ &= 2\left[S' - \frac{h_o}{2}\cdot\frac{S''}{S'}\right]. \end{aligned}\right\} \tag{138}$$

Die Bahngleichungen bis auf die erste Näherung:

Nach (127) lauten sie wie

$$\left.\begin{aligned} -\frac{dt}{d\tau} &= \frac{1}{\varepsilon_0}\left[eE\left(z+\frac{\varepsilon'}{eE}\right)-\frac{h_0}{2}\frac{eE}{w'}\right], \\ -\frac{dz}{cd\tau} &= \frac{c}{\varepsilon_0}\left[w'-\frac{h_0}{2}\cdot\frac{w''}{w'}\right], \\ -\frac{dx}{cd\tau} &= \frac{c}{\varepsilon_0}p_1, \quad -\frac{dy}{cd\tau} = \frac{c}{\varepsilon_0}p_2, \end{aligned}\right\} \tag{139}$$

wobei

$$w = S' = +\frac{1}{c}\sqrt{\left\{(eE)^2\left(z+\frac{\varepsilon'}{eE}\right)^2-(\varepsilon_0^2+c^2K^2)\right\}} = +\frac{dp_3}{dz}. \tag{140}$$

Bei nullter Näherung stimmen sie, wie man es merkt, mit (130) überein.

(*ii*) *Fassung* $[B_2(a),\ A_2(a)]$

Mit dem gleichen Ansatz, wie oben, nämlich, (117) hat man nach [35(*a*), I]

$$\left.\begin{aligned} \left[\left(h_0\frac{\partial}{c\partial t}-\frac{eV}{c}\right)^2-h_0^2\nabla^2-\frac{eh_0}{e}\sigma_z E\right]\begin{vmatrix}\chi_1\\ \chi_2\end{vmatrix} &= \frac{\varepsilon_0^2}{c^2}\begin{vmatrix}\chi_1\\ \chi_2\end{vmatrix}, \\ \left[\left(h_0\frac{\partial}{c\partial t}-\frac{eV}{c}\right)+h_0(\vec{\sigma}\vec{\nabla})\right]\begin{vmatrix}\chi_1\\ \chi_2\end{vmatrix} &= -\frac{\varepsilon_0}{c}\begin{vmatrix}\chi_1'\\ \chi_2'\end{vmatrix}. \end{aligned}\right\} \tag{141}$$

Mit der Setzung

$$(\chi_1, \chi_2;\ \chi_1', \chi_2') = (f, g;\ f', g')\ \exp\frac{\varepsilon t - xp_1 - yp_2}{h_0}, \tag{142}$$

ergeben sich daraus

$$\left.\begin{aligned} \left[\frac{d^2}{dz^2}+\left(\frac{\varepsilon_0^2+c^2K^2}{e^2h_0^2}+\frac{eE}{ch_0}\right\}-\left(\frac{eE}{ch_0}\right)^2\left(z+\frac{\varepsilon'}{eE}\right)^2\right]f &= 0, \\ \left[\frac{d^2}{dz^2}+\left\{\frac{\varepsilon_0^2+c^2K^2}{c^2h_0^2}-\frac{eE}{ch_0}\right\}-\left(\frac{eE}{ch_0}\right)^2\left(z+\frac{\varepsilon'}{eE}\right)^2\right]g &= 0. \end{aligned}\right\} \tag{143}$$

$$\left.\begin{aligned} -\frac{c}{\varepsilon_0}\left[\frac{eE}{c}\left(z+\frac{\varepsilon'}{eE}\right)f+h_0\frac{df}{dz}-(p_1-ip_2)g\right] &= f', \\ -\frac{c}{\varepsilon_0}\left[\frac{eE}{c}\left(z+\frac{\varepsilon'}{eE}\right)g-h_0\frac{dg}{dz}-(p_1+ip_2)f\right] &= g'. \end{aligned}\right\} \tag{144}$$

ε', K haben dieselben Bedeutungen wie in (122). Nach (143) u (144) findet man in Abhängigkeit von Hermiteschen Funktionen $\varphi_n(\xi)$, $\varphi_{n-1}(\xi)$

$$\left.\begin{aligned}
&f = \alpha\varphi_n(\xi).\ g = \beta\varphi_{n-1}(\xi),\\
&f' = \alpha'\varphi_{n-1}(\xi),\ g' = \beta'\varphi_n(\xi),\\
&\xi = \left(\frac{eE}{ch_0}\right)^{\frac{1}{2}}\left(z + \frac{\varepsilon}{eE}\right);\\
&\alpha' = -\frac{c}{\varepsilon_0}\left[\frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{c}\,\alpha - (p_1 - ip_1)\beta\right],\\
&\beta' = -\frac{c}{\varepsilon_0}\left[\frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{c}\,\beta - (p_1 + ip_2)\alpha\right],
\end{aligned}\right\} \tag{145}$$

wo gilt wie früher

$$(2n)\cdot\frac{eE}{ch_0} = \frac{\varepsilon_0^2 + c^2K^2}{c^2h_0^2}$$

Nun berechnet man nach [(48), I] die Bahngleichungen wie

$$\left.\begin{aligned}
&\frac{dt}{d\tau} = -\frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{2\varepsilon_0}\left(\frac{\varphi_{n-1}}{\varphi_n} + \frac{\varphi_n}{\varphi_{n-1}}\right),\\
&-\frac{dz}{cd\tau} = -\frac{\sqrt{(\varepsilon_0^2 + c^2K^2)}}{2\varepsilon_0}\left(\frac{\varphi_{n-1}}{\varphi_n} - \frac{\varphi_n}{\varphi_{n-1}}\right),\\
&-\frac{dx}{cd\tau} = \frac{c}{\varepsilon_0}p_1,\ -\frac{dy}{cd\tau} = \frac{c}{\varepsilon_0}p_2.
\end{aligned}\right\} \tag{146}$$

Die Klassische Bahn.

Mit Bezug auf [(B), I], aber nach gleichzeitigem Umschlagen des Vorzeichens von e und m_0 erhält man mit dem Ansatz (117)

$$\frac{\varepsilon_0^2}{c^2} = \left(\frac{\partial S}{c\partial t} - \frac{eV}{c}\right)^2 - \sum_x\left(\frac{\partial S}{\partial x}\right)^2. \tag{147}$$

Mit der Setzung

$$S = \varepsilon t - xp_1 - yp_2 - p_3(z) \tag{148}$$

hat man daraus

$$\left(\frac{dS}{dz}\right)^2 = \left(\frac{eE}{c}\right)^2\left(z + \frac{\varepsilon'}{eE}\right)^2 - \frac{\varepsilon_0^2 + c^2K^2}{c^2} = S'^2. \tag{149}$$

Nach [(A), I] lauten die klassischen Gleichungen der Bahn

$$\left.\begin{aligned}
&-\varepsilon_0\frac{dt}{d\tau} = \frac{\partial S}{\partial t} - eV = +eE\left(z + \frac{\varepsilon'}{eE}\right),\\
&\frac{\varepsilon_0}{c^2}\frac{dx}{d\tau} = \frac{\partial S}{\partial x} = -p_1,\ \frac{\varepsilon_0}{c^2}\frac{dy}{d\tau} = \frac{\partial S}{\partial y} = -p_2,\\
&\frac{\varepsilon_0}{c^2}\frac{dz}{d\tau} = \frac{\partial S}{\partial z} = -\sqrt{\left\{\left(\frac{eE}{c}\right)^2\left(z + \frac{\varepsilon'}{eE}\right)^2 - \frac{\varepsilon_0^2 + c^2K^2}{c^2}\right\}} = -\frac{dp_3}{dz}.
\end{aligned}\right\} \tag{150}$$

Reduktion von (143) bis auf did erste Näherung.

Verfährt man zwecks Näherung in dem vorigen Fall analoger Weise, so findet man zunächst noch das Gelten von (134) bei nullter Näherung, bis auf die erste Näherung aber

$$\left.\begin{aligned}\operatorname*{Limes}_{h_0\to 0}(h_0 \ln f)' &= \operatorname*{Limes}_{h_0\to 0}(h_0 \ln \varphi_n)' = S' + h_0\beta'_1,\\ \operatorname*{Limes}_{h_0\to 0}(h_0 \ln g)' &= \operatorname*{Limes}_{h_0\to 0}(h_0 \ln \varphi_{n-1})' = S' + h_0\alpha'_1,\end{aligned}\right\} \tag{151}$$

wo α'_1, β'_1 dieselben Ausdrücke wie in (135) haben.

Reduktion von (144) bis auf die erste Näherung.

Aus (144) leitet man im vorliegenden Falle zunächst (130) noch als gültig ab und daher bis auf die erste Näherung auch noch (138).

Die Bahngleichungen bis auf die erste Näherung.

Nach (148) lauten sie wie

$$\left.\begin{aligned}&-\frac{dt}{d\tau} = \frac{1}{\varepsilon_0}\left[eE\left(z + \frac{\varepsilon'}{eE}\right) + \frac{h_0}{2}\frac{eE}{w'}\right],\\ &-\frac{dz}{cd\tau} = \frac{c}{\varepsilon_0}\left[w' + \frac{h_0}{2}\frac{w''}{w'}\right],\\ &-\frac{dx}{cd\tau} = \frac{c}{\varepsilon_0}p_1,\quad -\frac{dy}{cd\tau} = \frac{c}{\varepsilon_0}p_2.\end{aligned}\right\} \tag{152}$$

wobei mit Rücksicht auf (134), (150)

$$w' = -S' = +\frac{1}{c}\sqrt{\left\{(eE)^2\left(z + \frac{\varepsilon'}{eE}\right)^2 - (\varepsilon_0^2 + c^2K^2)\right\}} = +\frac{dp_3}{dz}, \tag{153}$$

gesetzt worden ist.

Man sieht, dass die sich hieraus nach Abstreichen von h_0 bei nullter Näherung ergebenden Gleichungen mit (150) oder (130) übereinstimmen; die Glieder erster Ordnung rechts in (152) sind aber gegenüberden Gliedern erster Ordnung rechts in (139) von entgegengesetztem Vorzeichen.

4. Das Elektron in konstantem magnetischen Felde nach der Fassung (i) $[B_2, A_2]$ bzw. $[B_2(a), A_2(a)]$ des Teils I.

(i) Fassung $[B_2, A_2]$

Mit dem Ansatz

$$\left.\begin{aligned}\varphi &= 0 = A_z,\\ A_x &= -\tfrac{1}{2}Hy,\quad A_y = +\tfrac{1}{2}Hx,\end{aligned}\right\} \tag{154}$$

wo H die konstante Feldstärke in der z—Richtung bezeichnet, hat man nach [(35), I]

$$\left.\begin{aligned}&\left[h_0^2\frac{\partial^2}{c^2\partial t^2} - \left(h_0\frac{\partial}{\partial x} + \frac{eH}{2c}y\right)^2 - \left(h_0\frac{\partial}{\partial y} - \frac{eH}{2c}x\right)^2 - h_0^2\frac{\partial^2}{\partial z^2} + \frac{ieh_0}{c}\sigma_z H\right]\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix} = \frac{\varepsilon_0^2}{c^2}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix},\\ &\left[h_0\frac{\partial}{c\partial t} + \sigma_x\left(h_0\frac{\partial}{\partial x} + \frac{eH}{2c}y\right) + \sigma_y\left(h_0\frac{\partial}{\partial y} - \frac{eH}{2c}x\right) + \sigma_z h_0\frac{\partial}{\partial z}\right]\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix} = \frac{\varepsilon_0}{c}\begin{vmatrix}\psi'_1\\ \psi'_2\end{vmatrix}.\end{aligned}\right\} \tag{155}$$

Mit der Setzung

$$(\psi_1, \psi_2 ; \psi_1', \psi_2') = (f, g ; f', g') \exp \frac{-\varepsilon t + z p_3}{h_0}, \tag{156}$$

hat man hieraus zunächst

$$\left.\begin{aligned} &\left[\frac{\varepsilon^2}{c^2} - h_0^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{e^2H^2}{4c^2}(x^2+y^2) + h_0 \frac{eH}{c}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right. \\ &\qquad\qquad \left. - p_3^2 + \frac{i h_0 eH}{c}\sigma_z\right]\begin{vmatrix} f \\ g \end{vmatrix} = \frac{\varepsilon_0^2}{c^2}\begin{vmatrix} f \\ g \end{vmatrix}, \\ &\left[-\frac{\varepsilon}{c} + \sigma_x\left(h_0\frac{\partial}{\partial x} + \frac{eH}{2c}y\right) + \sigma_y\left(h_0\frac{\partial}{\partial y} - \frac{eH}{2c}x\right) + \sigma_z p_3\right]\begin{vmatrix} f \\ g \end{vmatrix} = \frac{\varepsilon_0}{c}\begin{vmatrix} f' \\ g' \end{vmatrix}. \end{aligned}\right\} \tag{157}$$

dann aber in Zylinderkoordinaten

$$\left.\begin{aligned} &\frac{K^2}{h_0^2}\begin{vmatrix} f \\ g \end{vmatrix} = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{r^2\partial\theta^2} + \frac{e^2H^2}{4c^2h_0^2}r^2 - \frac{eH}{ch_0}\frac{\partial}{\partial\theta} - i\frac{eH}{ch_0}\sigma_z\right]\begin{vmatrix} f \\ g \end{vmatrix}, \\ &\text{mit } K^2 = \frac{\varepsilon^2 - \varepsilon_0^2}{c^2} - p_3^2, \end{aligned}\right\} \tag{158}$$

$$\left.\begin{aligned} \frac{\varepsilon_0}{c}f' &= -\left(\frac{\varepsilon}{c} - p_3\right)f + h_0 e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right)g + \frac{ieH}{2c}e^{-i\theta}rg, \\ \frac{\varepsilon_0}{c}g' &= -\left(\frac{\varepsilon}{c} + p_3\right)g + h_0 e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right)f - \frac{ieH}{2c}e^{i\theta}rf. \end{aligned}\right\} \tag{159}$$

Setzt man zwecks Lösung weiter

$$f = au(r)\exp\left[\left(m - \frac{i}{2}\right)\theta\right], \quad g = bv(r)\exp\left[\left(m + \frac{i}{2}\right)\theta\right], \tag{160}$$

so ergeben sich aus (158)

$$\left.\begin{aligned} \frac{K^2}{h_0^2}u &= \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{(m - \frac{1}{2}i)^2}{r^2} + \frac{e^2H^2}{4c^2h_0^2}r^2 - \frac{eH}{ch_0}\left(m - \frac{i}{2}\right)\right]u, \\ \frac{K^2}{h_0^2}v &= \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{(m + \frac{1}{2}i)^2}{r^2} + \frac{e^2H^2}{4c^2h_0^2}r^2 - \frac{eH}{ch_0}\left(m - \frac{i}{2}\right)\right]v, \end{aligned}\right\} \tag{161}$$

und aus (159) mit dem Ansatz

$$\left.\begin{aligned} h_0\left[\frac{d}{dr} - \frac{i(m + \frac{1}{2}i)}{r}\right]v + \frac{ieH}{2c}rv &= K'u, \\ h_0\left[\frac{d}{dr} - \frac{i(m - \frac{1}{2}i)}{r}\right]u - \frac{ieH}{2c}ru &= K'v, \end{aligned}\right\} \tag{162}$$

$$\left.\begin{aligned} \frac{\varepsilon_0}{c}f' &= \left[-a\left(\frac{\varepsilon}{c} - p_3\right) + bK'\right]u\exp\left[\left(m - \frac{i}{2}\right)\theta\right], \\ \frac{\varepsilon_0}{c}g' &= \left[-b\left(\frac{\varepsilon}{c} + p_3\right) + aK'\right]v\exp\left[\left(m + \frac{i}{2}\right)\theta\right]. \end{aligned}\right\} \tag{163}$$

Aus (162) folgert man leicht

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{(m-\frac{1}{2}i)^2}{r^2}+\frac{e^2H^2}{4c^2h_0^2}r^2-\frac{eH}{ch_0}\left(m+\frac{i}{2}\right)\right]u=\frac{K'^2}{h_0^2}u. \tag{164}$$

Somit setzen wir mit Rucksicht auf (161)

$$K'=K, \tag{165}$$

und weiterhin

$$v=\bar{u}. \tag{166}$$

Zusammenstellend had man also endlich

$$\left.\begin{aligned}
&f=au\exp\left[\left(m-\frac{i}{2}\right)\theta\right],\quad g=b\bar{u}\exp\left[\left(m+\frac{i}{2}\right)\theta\right],\\
&f'=a'u\exp\left[\left(m-\frac{i}{2}\right)\theta\right],\quad g'=b'\bar{u}\exp\left[\left(m+\frac{i}{2}\right)\theta\right],\\
&a'=\frac{c}{\varepsilon_0}\left[-a\left(\frac{\varepsilon}{c}-p_3\right)+bK\right],\quad b'=\frac{c}{\varepsilon_0}\left[-b\left(\frac{\varepsilon}{c}+p_3\right)+aK\right].
\end{aligned}\right\} \tag{167}$$

Berechnet man nun nach [(39), I] die Bahngleichungen, so findet man

$$\left.\begin{aligned}
&\frac{dt}{d\tau}=\frac{\varepsilon}{\varepsilon_0},\quad \frac{dz}{cd\tau}=\frac{c}{\varepsilon_0}p_3,\\
&\frac{dx}{cd\tau}=\frac{c}{2\varepsilon_0}K\left(\frac{\bar{u}}{u}\exp(i\theta)+\frac{u}{\bar{u}}\exp(-i\theta)\right),\\
&\frac{dy}{cd\tau}=-\frac{c}{2\varepsilon_0}iK\left(\frac{\bar{u}}{u}\exp(i\theta)-\frac{u}{\bar{u}}\exp(-i\theta)\right),
\end{aligned}\right\} \tag{168}$$

Die klassische Bahn.

Nimmt man auf [(*B*), I] Bezug, so hat man mit dem Ansatz (154)

$$\left.\begin{aligned}
\frac{\varepsilon_0^2}{c^2}&=\left(\frac{\partial S}{c\partial t}\right)^2-\left(\frac{\partial S}{\partial x}+\frac{eH}{2c}y\right)^2-\left(\frac{\partial S}{\partial y}-\frac{eH}{2c}x\right)^2-\left(\frac{\partial S}{\partial z}\right)^2\\
&=\left(\frac{\partial S}{c\partial t}\right)^2-\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\frac{e^2H^2}{4c^2}(x^2+y^2)-\frac{eH}{c}\left(x\frac{\partial S}{\partial y}-y\frac{\partial S}{\partial x}\right)+\left(\frac{\partial S}{\partial z}\right)^2\right]\\
&=\left(\frac{\partial S}{c\partial t}\right)^2-\left(\frac{\partial S}{\partial z}\right)^2-\left[\left(\frac{\partial S}{\partial r}\right)^2+\left(\frac{\partial S}{r\partial\theta}\right)^2+\frac{e^2H^2}{4c^2}r^2-\frac{eH}{c}\cdot\frac{\partial S}{\partial\theta}\right],
\end{aligned}\right\} \tag{169}$$

nach Transformation auf Zylinderkoordinaten Mit der Setzung

$$S=-\varepsilon t+zp_3+a\,\theta+w(r), \tag{170}$$

hat man aus (169)

$$\left(\frac{\partial S}{\partial r}\right)^2=K^2-\frac{a^2}{r^2}+\frac{aeH}{c}-\frac{e^2H^2}{4c^2}r^2=\left(\frac{dw}{dr}\right)^2. \tag{171}$$

Nach [(*A*), I] lauten die klassischen Bahngleichungen wie

$$\left.\begin{aligned}
\varepsilon_0\frac{dt}{d\tau} &= \frac{\partial S}{\partial t} = -\varepsilon, \quad -\frac{\varepsilon_0}{c^2}\frac{dz}{d\tau} = \frac{\partial S}{\partial z} = p_3\\
-\frac{\varepsilon_0}{c^2}\frac{dx}{d\tau} &= \frac{\partial S}{\partial x} + \frac{eH}{2c}\,y,\\
&= w'\cos\theta - \left(\frac{a}{r} - \frac{eH}{2c}\,r\right)\sin\theta,\\
-\frac{\varepsilon_0}{c^2}\frac{dy}{d\tau} &= \frac{\partial S}{\partial y} - \frac{eH}{2c}\,x,\\
&= w'\sin\theta + \left(\frac{a}{r} - \frac{eH}{2c}\,r\right)\cos\theta.
\end{aligned}\right\} \tag{172}$$

Reduktion von (161) bis auf die erste Näherung.

Schreibt man (161) mit Rucksicht auf (166), wie folgt, hin

$$\left.\begin{aligned}
K^2 &= h_o^2\left[\left(\frac{u'}{u}\right)' + \left(\frac{u'}{u}\right)^2\right] + \frac{h_o^2}{r}\frac{u'}{u} + \frac{h_o^2(m^2 - im - \frac{1}{4})}{r^2} + \frac{e^2H^2}{4c^2}r^2 - \frac{eH}{c}\,h_o\left(m + \frac{i}{2}\right),\\
K^2 &= h_o^2\left[\left(\frac{\bar u'}{\bar u}\right)' + \left(\frac{\bar u'}{\bar u}\right)^2\right] + \frac{h_o^2}{r}\frac{\bar u'}{\bar u} + \frac{h_o^2(m^2 + im - \frac{1}{4})}{r^2} + \frac{e^2H^2}{4c^2}r^2 - \frac{eH}{c}\,h_o\left(m - \frac{i}{2}\right),
\end{aligned}\right\} \tag{173}$$

und fuhrt dort zwecks Näherung

$$\left.\begin{aligned}
&\operatorname*{Limes}_{h_o\to 0}\,(h_o \ln u) = S' + h_o\alpha_1' + h_o^2\alpha_2' + \dots\,,\\
&\operatorname*{Limes}_{h_o\to 0}\,(h_o \ln \bar u)' = S' + h_o\bar\alpha_1' + h_o^2\bar\alpha_2' + \dots\,\cdot\\
&\operatorname*{Limes}_{h_o\to 0}\,(h_o m) = a,
\end{aligned}\right\} \tag{174}$$

ein, so erhält man in nullter Näherung

$$S'^2 + \frac{a^2}{r^2} - \frac{eH}{c}\,a + \frac{e^2H^2}{4c^2}r^2 = K^2. \tag{175}$$

in erster Näherung aber

$$\left.\begin{aligned}
\alpha_1' + \frac{1}{2r} &= -\tfrac{1}{2}\left[\frac{S''}{S'} - \frac{i}{S'}\left(\frac{a}{r^2} + \frac{eH}{2c}\right)\right],\\
\alpha_1' + \frac{1}{2r} &= -\tfrac{1}{2}\left[\frac{S''}{S'} + \frac{i}{S}\left(\frac{a}{r^2} + \frac{eH}{2c}\right)\right].
\end{aligned}\right\} \tag{176}$$

Reduktion von (162).

Aus (162) hat man mittels (174)

$$\left.\begin{aligned}
(S' + h_o\alpha_1' + h_o^2\alpha_2' + \dots) + \frac{ia}{r} + \frac{h_o}{2r} - \frac{ieH}{2c}r &= K\frac{\bar u}{u},\\
(S' + h_o\bar\alpha_1' + h_o^2\bar\alpha_2' + \dots) - \frac{ia}{r} + \frac{h_o}{2r} + \frac{ieH}{2c}\,r &= K\frac{u}{\bar u}.
\end{aligned}\right\} \tag{177}$$

Daher folgen im Limes ($h_0 \rightarrow 0$) bis auf die erste Näherung

$$\left.\begin{aligned}
&K\left(\frac{\bar{u}}{u}\exp(i\theta)+\frac{u}{\bar{u}}\exp(-i\theta)\right)_L = 2\left[S'\cos\theta-\left(\frac{a}{r}-\frac{eH}{2c}r\right)\sin\theta\right] \\
&\qquad -h_0\frac{(a+(eH/2c)r^2)}{2r^2S'^2}\cdot 2\left[\left(\frac{a}{r}-\frac{eH}{2c}r\right)\cos\theta+S'\sin\theta\right], \\
&-iK\left(\frac{\bar{u}}{u}\exp(i\theta)-\frac{u}{\bar{u}}\exp(-i\theta)\right)_L = 2\left[S'\sin\theta+\left(\frac{a}{r}-\frac{eH}{2c}r\right)\cos\theta\right] \\
&\qquad +h_0\frac{(a+(eH/2c)r^2)}{2r^2S'^2}\cdot 2\left[S'\cos\theta-\left(\frac{a}{r}-\frac{eH}{2c}r\right)\sin\theta\right].
\end{aligned}\right\} \tag{178}$$

Bis auf die erste Näherung lauten also nach (168) die Bahngleichungen als

$$\left.\begin{aligned}
&-\frac{dt}{d\tau}=\frac{\varepsilon}{\varepsilon_0},\quad -\frac{dz}{cd\tau}=\frac{c}{\varepsilon_0}p_3, \\
&-\frac{dx}{cd\tau}=\frac{c}{\varepsilon_0}\left[\omega'\cos\theta-\left(\frac{a}{r}-\frac{eH}{2c}r\right)\sin\theta\right] \\
&\qquad -\frac{c}{\varepsilon_0}h_0\frac{(a+(eH/2c)r^2)}{2r^2w'^2}\left[\left(\frac{a}{r}-\frac{eH}{2c}r\right)\cos\theta+w'\sin\theta\right], \\
&-\frac{dy}{cd\tau}=\frac{c}{\varepsilon_0}\left[w'\sin\theta+\left(\frac{a}{r}-\frac{eH}{2c}r\right)\cos\theta\right] \\
&\qquad +\frac{c}{\varepsilon_0}h_0\frac{(a+(eH/2c)r^2)}{2r^2w'^2}\left[w'\cos\theta-\left(\frac{a}{r}-\frac{eH}{2c}r\right)\sin\theta\right].
\end{aligned}\right\} \tag{179}$$

Dabei versteht man mit Rücksicht auf (170). (171)

$$w'=+\sqrt{\left(K^2-\frac{a^2}{r^2}+\frac{aeH}{c}-\frac{e^2H^2}{4c^2}r^2\right)}=+\frac{\partial S}{\partial r}. \tag{180}$$

Mit Abstreichen von h_0 bei nullter Näherung gehen also (179) in (172) über.

(ii) *Fassung* $[B_2(a),\ A_2(a)]$

Bezieht man sich auf [35(a), I], so hat man mit dem gleichen Ansatz (154), wie oben

$$\left.\begin{aligned}
&\left[h_0^2\frac{\partial^2}{c^2\partial t^2}-\left(h_0\frac{\partial}{\partial x}-\frac{eH}{2c}y\right)^2-\left(h_0\frac{\partial}{\partial y}+\frac{eH}{2c}x\right)^2-h_0^2\frac{\partial^2}{\partial z^2}-\frac{ieh}{c}\sigma_z H\right]\begin{vmatrix}\chi_1\\ \chi_2\end{vmatrix}=\frac{\varepsilon_0^2}{c^2}\begin{vmatrix}\chi_1\\ \chi_2\end{vmatrix}, \\
&\left[h_0\frac{\partial}{c\partial t}+\sigma_x\left(h_0\frac{\partial}{\partial x}-\frac{eH}{2c}y\right)+\sigma_y\left(h_0\frac{\partial}{\partial y}+\frac{eH}{2c}x\right)+\sigma_z h_0\frac{\partial}{\partial z}\right]\begin{vmatrix}\chi_1\\ \chi_2\end{vmatrix}=-\frac{\varepsilon_0}{c}\begin{vmatrix}\chi_1'\\ \chi_2'\end{vmatrix}.
\end{aligned}\right\} \tag{181}$$

Mit der Setzung

$$(\chi_1,\ \chi_2,,\ \chi_1',\ \chi_2')=(f,\ g\ ;\ f',\ g')\exp\frac{\varepsilon t-zp_3}{h_0}, \tag{182}$$

hat man daraus zunächst

$$\left.\begin{aligned} &\left[\frac{\varepsilon^2}{c^2}-h_0^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{e^2H^2}{4c^2}(x^2+y^2)-h_0\,\frac{eH}{c}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)-p_3^2-\frac{ieh_0}{c}\sigma_z H\right]\left|\begin{matrix} f \\ g \end{matrix}\right|=\frac{\varepsilon_0^2}{c^2}\left|\begin{matrix} f \\ g \end{matrix}\right|, \\ &\left[\frac{\varepsilon}{c}+\sigma_x\left(h_0\frac{\partial}{\partial x}-\frac{eH}{2c}\,y\right)+\sigma_y\left(h_0\frac{\partial}{\partial y}+\frac{eH}{2c}\,x\right)-\sigma_z p_3\right]\left|\begin{matrix} f \\ g \end{matrix}\right|=-\frac{\varepsilon_0}{c}\left|\begin{matrix} f' \\ g' \end{matrix}\right|. \end{aligned}\right\} \quad (183)$$

dann aber in Zylinderkoordinaten

$$\left.\begin{aligned} &\frac{K^2}{h_0^2}\left|\begin{matrix} f \\ g \end{matrix}\right|=\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{r^2\partial\theta^2}+\frac{e^2H^2}{4c^2h_0^2}\,r^2+\frac{eH}{ch_0}\,\frac{\partial}{\partial\theta}+i\,\frac{eH}{ch_0}\,\sigma_z\right]\left|\begin{matrix} f \\ g \end{matrix}\right|, \\ &\text{mit } K^2=\frac{\varepsilon^2-\varepsilon_0^2}{c^2}-p_3^2, \end{aligned}\right\} \quad (184)$$

$$\left.\begin{aligned} -\frac{\varepsilon_0}{c}f' &=\left(\frac{\varepsilon}{c}-p_3\right)f+h_0e^{-i\theta}\left(\frac{\partial}{\partial r}-\frac{i}{r}\,\frac{\partial}{\partial\theta}\right)g-\frac{ieH}{2c}\,e^{-i\theta}rg, \\ -\frac{\varepsilon_0}{c}\,g' &=\left(\frac{\varepsilon}{c}+p_3\right)g+h_0e^{i\theta}\left(\frac{\partial}{\partial r}+\frac{i}{r}\,\frac{\partial}{\partial\theta}\right)f+\frac{ieH}{2c}\,e^{i\theta}rf. \end{aligned}\right\} \quad (185)$$

Setzt man zwecks Lösung

$$f=AU(r)\exp\left[-\left(m+\frac{i}{2}\right)\theta\right],\quad g=B\,V(r)\exp\left[-\left(m-\frac{i}{2}\right)\theta\right], \quad (186)$$

so erhält man aus (184)

$$\left.\begin{aligned} \frac{K^2}{h_0^2}\,U &=\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{(m+\frac{1}{2}i)^2}{r^2}+\frac{e^2H^2}{4c^2h_0^2}r^2-\frac{eH}{ch_0}\left(m-\frac{i}{2}\right)\right]U, \\ \frac{K^2}{h_0^2}\,V &=\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{(m-\frac{1}{2}i)^2}{r^2}+\frac{e^2H^2}{4c^2h_0^2}r^2-\frac{eH}{ch_0}\left(m+\frac{i}{2}\right)\right]V, \end{aligned}\right\} \quad (187)$$

und aus (185) mit dem Ansatz

$$\left.\begin{aligned} h_0\left[\frac{d}{dr}+\frac{i(m-\frac{1}{2}i)}{r}\right]V-\frac{ieH}{2c}\,rV &= K'U, \\ h_0\left[\frac{d}{dr}-\frac{i(m+\frac{1}{2}i)}{r}\right]U+\frac{ieH}{2c}\,rU &= K'V, \end{aligned}\right\} \quad (188)$$

$$\left.\begin{aligned} -\frac{\varepsilon_0}{c}f' &=\left[A\left(\frac{\varepsilon}{c}-p_3\right)+BK'\right]U\exp\left[-\left(m+\frac{i}{2}\right)\theta\right], \\ -\frac{\varepsilon}{c}g' &=\left[B\left(\frac{\varepsilon}{c}+p_3\right)+AK'\right]V\exp\left[-\left(m-\frac{i}{2}\right)\theta\right]. \end{aligned}\right\} \quad (189)$$

Aus (188) folgert man leicht

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{(m+\frac{1}{2}i)^2}{r^2}+\frac{e^2H^2}{4c^2h_0^2}\,r^2-\frac{eH}{ch_0}\left(m-\frac{i}{2}\right)\right]U=\frac{K'^2}{h_0^2}\,U, \quad (190)$$

wonach man mit Rücksicht auf (187)

$$K' = K, \tag{191}$$

setzt. Die Gegenuberstellung von (188) mit (162) ergibt fernerhin

$$\left.\begin{aligned} U &= \bar{u}, \\ V &= u. \end{aligned}\right\} \tag{192}$$

Die Lösungsergebnisse stellt man demnach, wie folgt, zusammen:

$$\left.\begin{aligned} &f = A\bar{u}\exp\left[-\left(m+\frac{i}{2}\right)\theta\right],\ g = Bu\exp\left[-\left(m-\frac{i}{2}\right)\theta\right], \\ &f' = A\bar{u}\exp\left[-\left(m+\frac{i}{2}\right)\theta\right],\ g' = B'u\exp\left[-\left(m-\frac{i}{2}\right)\theta\right], \\ &A' = -\frac{c}{\varepsilon_0}\left[A\left(\frac{\varepsilon}{c}-p_3\right)+BK\right], \\ &B' = -\frac{c}{\varepsilon_0}\left[B\left(\frac{\varepsilon}{c}+p_3\right)+AK\right]. \end{aligned}\right\} \tag{193}$$

Berechnet man nun nach [(43), I] die Bahngleichungen, so findet man

$$\left.\begin{aligned} &-\frac{dt}{d\tau} = \frac{\varepsilon}{\varepsilon_0},\ -\frac{dz}{cd\tau} = \frac{c}{\varepsilon_0}p_3, \\ &-\frac{dx}{cd\tau} = -\frac{c}{2\varepsilon_0}K\left(\frac{\bar{u}}{u}\exp(-i\theta)+\frac{u}{\bar{u}}\exp(i\theta)\right), \\ &-\frac{dy}{cd\tau} = -\frac{c}{2\varepsilon_0}iK\left(\frac{\bar{u}}{u}\exp(-i\theta)-\frac{u}{\bar{u}}\exp(i\theta)\right). \end{aligned}\right\} \tag{194}$$

Die klassische Bahn

Mit Bezugnahme auf [(B), I] aber nach gleichzeitigen Umschlagen des Vorzeichens von e und m_0 erhält man mit dem Ansatz (154)

$$\left.\begin{aligned} \frac{\varepsilon_0^2}{c^2} &= \left(\frac{\partial S}{c\partial t}\right)^2-\left(\frac{\partial S}{\partial x}-\frac{eH}{2c}y\right)^2-\left(\frac{\partial S}{\partial y}+\frac{eH}{2c}x\right)^2-\left(\frac{\partial S}{\partial z}\right)^2 \\ &= \left(\frac{\partial S}{c\partial t}\right)^2-\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\frac{e^2H^2}{4c^4}(x^2+y^2)+\frac{eH}{c}\left(x\frac{\partial S}{\partial y}-y\frac{\partial S}{\partial x}\right)+\left(\frac{\partial S}{\partial z}\right)^2\right] \\ &= \left(\frac{\partial S}{c\partial t}\right)^2-\left(\frac{\partial S}{\partial z}\right)^2-\left[\left(\frac{\partial S}{\partial r}\right)^2+\left(\frac{\partial S}{r\partial\theta}\right)^2+\frac{e^2H^2}{4c^2}r^2+\frac{eH}{c}\frac{\partial S}{\partial\theta}\right], \end{aligned}\right\} \tag{195}$$

nach Transformation auf Zylinderkoordinaten.

Mit der Setzung

$$S = \varepsilon t - zp_3 - a\theta - w(r), \tag{196}$$

hat man aus (195)

$$\left(\frac{\partial S}{\partial r}\right)^2 = K^2-\frac{a^2}{r^2}+\frac{aeH}{c}-\frac{e^2H^2}{4c^2}r^2 = \left(\frac{dw}{dr}\right)^2. \tag{197}$$

Nach [(*A*), I] lauten die klassischen Bahngleichungen wie

$$\left.\begin{aligned} -\varepsilon_0\frac{dt}{d\tau} &= \frac{\partial S}{\partial t} = \varepsilon, \quad \frac{\varepsilon_0}{c^2}\frac{dz}{d\tau} = \frac{\partial S}{\partial z} = -p_3, \\ \frac{\varepsilon_0}{c^2}\frac{dx}{d\tau} &= \frac{\partial S}{\partial x} - \frac{eH}{2c}y = -w'\cos\theta + \left(\frac{a}{r} - \frac{eH}{2c}r\right)\sin\theta, \\ \frac{\varepsilon_0}{c^2}\frac{dy}{d\tau} &= \frac{\partial S}{\partial y} + \frac{eH}{2c}x = -w'\sin\theta - \left(\frac{a}{r} - \frac{eH}{2c}r\right)\cos\theta. \end{aligned}\right\} \tag{198}$$

Reduktion von (190) bzw. (188) bis auf die erste Näherung.

Man sieht nun, dass wegen (192) alle Gleichungen von (178) bis (177) fur den vorliegenden Fall verwendet werden dürfen ebenso wie für den früheren. Daher findet man nach (177) im Limes ($h_o \to 0$) und bis auf die erste Näherung

$$\left.\begin{aligned} K\left(\frac{\bar{u}}{u}\exp(-i\theta) + \frac{u}{\bar{u}}\exp(i\theta)\right)_L &= 2\left[S'\cos\theta + \left(\frac{a}{r} - \frac{eH}{2c}r\right)\sin\theta\right] \\ &\quad - h_o\frac{a + (eH/2c)r^2}{2r^2S'^2}\cdot 2\left[\left(\frac{a}{r} - \frac{eH}{2c}r\right)\cos\theta - S'\sin\theta\right], \\ iK\left(\frac{\bar{u}}{u}\exp(-i\theta) - \frac{u}{\bar{u}}\exp(i\theta)\right)_L &= 2\left[S'\sin\theta - \left(\frac{a}{r} - \frac{eH}{2c}r\right)\cos\theta\right] \\ &\quad - h_o\frac{a + (eH/2c)r^2}{2r^3S'^2}\cdot 2\left[\left(\frac{a}{r} - \frac{eH}{2c}r\right)\sin\theta + S'\cos\theta\right]. \end{aligned}\right\} \tag{199}$$

Setzt man nun mit Rucksicht auf (196), (197)

$$w' = +\sqrt{\left(K^2 - \frac{a^2}{r^2} + \frac{aeH}{c} - \frac{e^2H^2}{4c^2}r^2\right)} = -\frac{\partial S}{\partial r}, \tag{200}$$

so lauten nach (194) die Bahngleichungen bis auf die erste Näherung als

$$\left.\begin{aligned} -\frac{dt}{d\tau} &= \frac{\varepsilon}{\varepsilon_0}, \quad -\frac{dz}{cd\tau} = \frac{c}{\varepsilon_0}p_3, \\ -\frac{dx}{cd\tau} &= \frac{c}{\varepsilon_0}\left[w'\cos\theta - \left(\frac{a}{r} - \frac{eH}{2c}r\right)\sin\theta\right] \\ &\quad + \frac{c}{\varepsilon_0}h_o\frac{a + (eH/2c)r^2}{2r^2w'^2}\left[\left(\frac{a}{r} - \frac{eH}{2c}r\right)\cos\theta + w'\sin\theta\right], \\ -\frac{dy}{cd\tau} &= \frac{c}{\varepsilon_0}\left[w'\sin\theta + \left(\frac{a}{r} - \frac{eH}{2c}r\right)\cos\theta\right] \\ &\quad + \frac{c}{\varepsilon_0}h_o\frac{a + (eH/2c)r^2}{2r^2w'^2}\left[\left(\frac{a}{r} - \frac{eH}{2c}r\right)\sin\theta - w'\cos\theta\right]. \end{aligned}\right\} \tag{201}$$

Mit Abstreichen von h_0, also bei unllter Näherung, gehen (201) in (188) oder (127) über; aber die Glieder erster Ordnung in (201) sind gegenüber den Gliedern erster Ordnung in (179) von entgegengesetztem Vorzeichen.

Das feldfreie Elektron nach der Fassung (*i*) $[B_2, A_2]$ *bzw.* $[B_2(a), A_2(a)]$ *des Teils I.*

(*i*) Fassung $[B_2, A_2]$.

Mit dem Ansatz

$$\varphi = 0 = A_x = A_y = A_z, \tag{202}$$

hat man nach [(85), I]

$$\left.\begin{aligned} \left[h_0^2 \frac{\partial^2}{c^2\partial t^2} - h_0^2\nabla^2\right]\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix} &= \frac{\varepsilon_0^2}{c^2}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}, \\ \left[h_0 \frac{\partial}{c\partial t} + h_0(\vec{\sigma}\vec{\nabla})\right]\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix} &= \frac{\varepsilon_0}{c}\begin{vmatrix}\psi_1'\\ \psi_2'\end{vmatrix}. \end{aligned}\right\} \tag{203}$$

Mit der Setzung

$$(\psi_1, \psi_2; \psi_1', \psi_2') = (a, b, a', b') \exp \frac{-\varepsilon t + xp_1 + yp_2 + zp_3}{h_0}, \tag{204}$$

hat man daher

$$\frac{\varepsilon^2}{c^2} - (p_1^2 + p_2^2 + p_3^2) = \frac{\varepsilon_0^2}{c^2}, \tag{205}$$

$$\left.\begin{aligned} a' &= \frac{c}{\varepsilon_0}\left[-\left(\frac{\varepsilon}{c} - p_3\right)a + (p_1 - ip_2)b\right], \\ b' &= \frac{c}{\varepsilon_0}\left[-\left(\frac{\varepsilon}{c} + p_3\right)b + (p_1 + ip_2)a\right]. \end{aligned}\right\} \tag{206}$$

Nach [(39). I] findet man die Bahngleichungen als

$$\left.\begin{aligned} \frac{dt}{d\tau} &= -\frac{\varepsilon}{\varepsilon_0}, & \frac{dz}{cd\tau} &= -\frac{c}{\varepsilon_0}p_3, \\ \frac{dx}{cd\tau} &= -\frac{c}{\varepsilon_0}p_1, & \frac{dy}{cd\tau} &= -\frac{c}{\varepsilon_0}p_2. \end{aligned}\right\} \tag{207}$$

Die klassische Bahn,

Mit Bezugnahme auf [(*B*), I] hat man

$$\frac{\varepsilon_0^2}{c^2} = \left[\left(\frac{\partial S}{c\partial t}\right)^2 - \sum_x\left(\frac{\partial S}{\partial x}\right)^2\right]. \tag{208}$$

Mit der Setzung

$$S = -\varepsilon t + xp_1 + yp_2 + zp_3, \tag{209}$$

folgt hieraus

$$\frac{\varepsilon_0^2}{c^2} = \frac{\varepsilon^2}{c^2} - (p_1^2 + p_2^2 + p_3^2),$$

und nach $[(A), 1]$ lauten die klassischen Gleichungen dann

$$\left.\begin{aligned} \varepsilon_0 \frac{dt}{d\tau} = -\varepsilon,\ p_3 = -\frac{\varepsilon_0}{c}\, \frac{dz}{cd\tau}, \\ p_1 = -\frac{\varepsilon_0}{e} \cdot \frac{dx}{cd\tau},\ p_2 = -\frac{\varepsilon_0}{c}\, \frac{dy}{cd\tau} \end{aligned}\right\} \tag{210}$$

Man merkt. dass (210) mit (207) gleich lauten.

(*ii*) *Fassung* $[B_2(a), A_2(a)]$.

Nach $[35(a), \mathrm{I}]$ hat man mit dem gleichen Ansatz (202), wie oben,

$$\left.\begin{aligned} \left[h_0 \frac{\partial^2}{c^2 \partial t^2} - h_0^2 \nabla^2\right] \begin{vmatrix} \chi_1 \\ \chi_2 \end{vmatrix} = \frac{\varepsilon_0^{3}}{c^2} \begin{vmatrix} \chi_1 \\ \chi_2 \end{vmatrix}, \\ \left[h_0 \frac{\partial}{c\partial t} + h_0 (\vec{\sigma} \vec{\nabla})\right] \begin{vmatrix} \chi_1 \\ \chi_2 \end{vmatrix} = -\frac{\varepsilon_0}{c} \begin{vmatrix} \chi_1' \\ \chi_2' \end{vmatrix}. \end{aligned}\right\} \tag{211}$$

Mit der Setzung

$$(\chi_1, \chi_2; \chi_1', \chi_2') = (\alpha, \beta; \alpha', \beta') \exp \frac{\varepsilon t - x p_1 - y p_2 - z p_3}{h_0}, \tag{212}$$

hat man daher

$$\left.\begin{aligned} \frac{\varepsilon^2}{c^2} - (p_1^2 + p_2^2 + p_3^2) &= \frac{\varepsilon_0^2}{c^2}, \\ \alpha' &= -\frac{c}{\varepsilon_0}\left[\left(\frac{\varepsilon}{c} - p_3\right)\alpha - (p_1 - ip_2)\beta\right], \\ \beta' &= -\frac{c}{\varepsilon_0}\left[\left(\frac{\varepsilon}{c} + p_3\right)\beta - (p_1 + ip_2)\alpha\right]. \end{aligned}\right\} \tag{213}$$

Nach $[(43), \mathrm{I}]$ ergeben sich die Bahngleichungen als

$$\left.\begin{aligned} \frac{dt}{d\tau} = -\frac{\varepsilon}{\varepsilon_0},\ \frac{dz}{cd\tau} = -\frac{c}{\varepsilon_0} p_3, \\ \frac{dx}{cd\tau} = -\frac{c}{\varepsilon_0} p_1,\ \frac{dy}{cd\tau} = -\frac{c}{\varepsilon_0} p_2. \end{aligned}\right\} \tag{214}$$

also von gleicher Gestalt wie (207).

Die klassische Bahn.

Mit Bezugnahme auf $[(B), \mathrm{I}]$, doch nach Umschlagen des Vorzeichens von ε_0 hat man wieder

$$\frac{\varepsilon_0^2}{c^2} = \left[\left(\frac{\partial S}{c\partial t}\right)^2 - \sum_x \left(\frac{\partial S}{\partial x}\right)^2\right].$$

Mit der Setzung

$$S = \varepsilon t - x p_1 - y p_2 - z p_3, \tag{215}$$

folgt hieraus

$$\frac{\varepsilon_0^2}{c^2} = \frac{\varepsilon^2}{c^2} - (p_1^2+p_2^2+p_3^2),$$

und nach [(*A*), *I*], doch mit Umänderung des Vorzeichens von ε_0, lauten die klassischen Bahngleichungen als

$$\left.\begin{aligned} -\varepsilon_0\frac{dt}{d\tau} &= \varepsilon, \quad -p_3 = \frac{\varepsilon_0}{c}\frac{dz}{c\,d\tau}, \\ -p_1 &= \frac{\varepsilon_0}{c}\frac{dx}{c\,d\tau}, \quad -p_2 = \frac{\varepsilon_0}{c}\frac{dy}{c\,d\tau}. \end{aligned}\right\} \tag{216}$$

Es lauten (216) gleich mit (214) wie auch mit (210) und (207).

6. Schlussbemerkungen.

Am Schlusse wollen wir die Aufmerksamkeit des Lesers auf diejenigen Ergebnisse der vorliegenden Abhandlung lenken, welche uns besonders beachtenswert erscheinen. Für das atomare Elektron wissen wir, nämlich, dessen Energie im *n*-ten Energiezustande durch die Formel [(66), (72)]

$$\varepsilon_n = \varepsilon_0\left\{1 - \frac{\gamma^2Z^2}{(s-1+\sqrt{\{(l+1)^2+\gamma^2Z^2\}})^2}\right\}^{-\frac{1}{2}},$$

auszudrücken. Stellt man diese der entsprechenden Dirac' schen Formel, nämlich,

$$\varepsilon_n = \varepsilon_0\left\{1 + \frac{\gamma^2Z^2}{(s-1+\sqrt{\{(l+1)^2-\gamma^2Z^2\}})^2}\right\}^{-\frac{1}{2}}, \tag{217}$$

gegenüber, so fallen uns die nachstehenden Unterschiede frappant auf. Bei Dirac ist: $\varepsilon_n < \varepsilon_0$, *bei uns aber*: $\varepsilon_n > \varepsilon_0$. Bei Dirac besteht die Möglichkeit eventuellen Verschwindens von $\sqrt{\{(l+1)^2-\gamma^2Z^2\}}$, falls $l = 0$, *bei uns kann aber der entsprechende Ausdruck*: $\sqrt{\{(l+1)^2+\gamma^2Z^2\}}$ *nie verschwinden.* Bei uns sind die Bahnen des atomaren Elektrons sämtlich radial gerichtet und von Ellipsenbahnen ist keine Rede. Hierfür blicke man auf (74), (79) bzw. 74(*a*), 79(*a*) zurück. Darüber hinaus tritt uns das Eigentümlichste der vorgelegten Theorie darin entgegen, dass sämtliche Quantenbahnen verdoppelt auftreten. Man stelle hierfür 74(*a*) bzw. 79(*a*) gegenüber (74) bzw. (79). Will man die Verdoppelungsverhältnisse bis auf die erste Näherung verfolgen, so hat man dazu (114) bzw. (116) an die Seite von (113) bzw. (115) zu stellen. Man sieht aber danach, dass bei nullter Näherung keine Doppelheit auftritt und die Bahn dann mit der klassischen zusammenfällt.

Dieselbe Verdoppelung von Quantenbahnen begegnet uns bei den Fällen (*i*) des Elektrons in konstantem elektrischen Felde und (*ii*) des Elektrons in konstantem magnetischen Felde. Im ersten Falle stelle man (127), (146) einander gegenüber für die unverkürzten Quantenbahnen, aber (139), (152) für die Quantenbahnen bis auf die erste Näherung. Im zweiten Falle vergleiche man (168), (194) miteinander für die unverkürzten Quantenbahnen, aber (179) mit (201) für die Quantenbahnen bis auf die erste Näherung.

In beiden Fällen verschwindet die Verdoppelung, falls man sich auf die nullte Näherung beschränkt und die Bahnen nehmen dann die klassische Gestalt an. Einzig beim feldfreien Elektron liefert die Theorie keine Verdoppelung von der Quantenbahn, welche auch mit der klassischen Bahn zusammenfällt.

In der klassischen Hamilton-Jakobischen Dynamik erscheinen die Bahngleichungen des negativen Elektrons, wie man es merkt, durchweg mit einem negativen Vorzeichen der Eigenzeit $d\tau$ versehen. Dieser Zug der klassischen Dynamik setzt sich durch die hier entworfene, eben auf der klassischen Hamilton Jakobischen Dynamik gegründete Quantendynamik fort. Für die phenomenale Bahn, *d.h.* die von der phenomenalen Zeit t abhängige Bahn, um die nur der messende Physiker zu schaffen hat, hat es aber keinen Belang, da stets $dt/d\tau$ auch negativ ist. Im Falle des feldfreien Elektrons, zum Beispiel, sind die Gleichungen der phenomenalen Bahn nach (210) oder (216)

$$\left(\frac{dx}{cdt}, \frac{dy}{cdt}, \frac{dz}{cdt}\right) = \frac{c}{\varepsilon}(p_1, p_2, p_3). \tag{218}$$

Man überzeugt sich ohne allzu viele Mühe, dass in den entsprechenden klassischen Gleichungen des positiven Elektrons das Vorzeichen von $d\tau$ positiv wird. Was sich am Tatbestand—das positive Vorzeichen von $d\tau$ für das positive Elektron, das negative Vorzeichen hingegen für das negative Elektron—physikalisch äussern will, ist uns noch nicht klar.

Nachtrag. Es leuchtet uns nachträglich ein, dass sich das Versagen der Kausalität bei der oben ausgearbeiteten Quantendynamik als mit einer Unsicherheit der Anfangsgeschwindigkeit verbunden ansehen lässt. Man sieht das am einfachsten an die oben behandelten Beispiele ein. Blickt man etwa beim Beispiele des konstanten elektrischen Feldes auf die Gleichungen (139) bzw. (152), so merkt man, dass zu Anfang der quantendynamischen Laufbahn die Geschwindigkeitskomponenten seien *entweder nach (139)*

$$\left.\begin{aligned}
\frac{dt}{d\tau} &= -\frac{eE}{\varepsilon_0}\left(z+\frac{\varepsilon'}{eE}\right)+\frac{h_0}{2\varepsilon_0}\cdot\frac{eE}{w'},\\
\frac{dz}{cd\tau} &= -\frac{c}{\varepsilon_0}w'+\frac{ch_0}{2\varepsilon_0}\cdot\frac{w''}{w'},\\
\frac{dx}{cd\tau} &= -\frac{c}{\varepsilon_0}p_1,\ \frac{dy}{cd\tau} = -\frac{c}{\varepsilon_0}p_2,
\end{aligned}\right\}$$

oder nach (152)

$$\left.\begin{aligned}
\frac{dt}{d\tau} &= -\frac{eE}{\varepsilon_0}\left(z+\frac{\varepsilon'}{cE}\right)-\frac{h_0}{2\varepsilon_0}\cdot\frac{eE}{w'},\\
\frac{dz}{cdt} &= -\frac{c}{\varepsilon_0}w'-\frac{ch_0}{2\varepsilon_0}\cdot\frac{w''}{w'},\\
\frac{dx}{cd\tau} &= -\frac{c}{\varepsilon_0}p_1,\ \frac{dy}{cd\tau} = -\frac{c}{\varepsilon_0}p_2.
\end{aligned}\right\}$$

Ist die vorangehende Bahn klassisch erfolgt, so sind die Geschwindigkeitskomponenten an deren Ende nach (130) oder (150)

$$\left.\begin{aligned} \frac{dt}{d\tau} &= -\frac{eE}{\varepsilon_0}\left(z+\frac{\varepsilon'}{eE}\right),\\ \frac{dz}{cd\tau} &= -\frac{c}{\varepsilon_0}\cdot w',\\ \frac{dx}{cd\tau} &= -\frac{c}{\varepsilon_0}p_1,\ \frac{dy}{cd\tau} = -\frac{c}{\varepsilon_0}p_2. \end{aligned}\right\}$$

Die Anfangsgeschwindigkeiten der quantendynamischen Bahn sind danach als unsicher zu meinen und die Unsicherheit betrifft nur die Komponenten $dt/d\tau$, $dz/cd\tau$ und im Masse von $\pm\frac{h_0}{2\varepsilon_0}\frac{eE}{w'}$ bzw. $\pm\frac{ch_0}{2\varepsilon_0}\cdot\frac{w''}{w'}$. Eine Aussage gleicher Art gilt offenbar auch in den übrigen Fällen.

Referat

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